Advanced Topics in Geometry F (MTH.B502)

Riemannian metrics

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Riemannian manifolds

Definition

A Riemannian metric (resp. pseudo Riemannian metric) is a section $g \in \Gamma(S(T^*M \otimes T^*M))$ such that the quadratic form g_p on T_pM is positive definite inner product (resp. non-degenerate inner product) on T_pM. A pair (M, g) of a manifold M and a (pseudo) Riemannian metric g is called a (pseudo) Riemannian manifold.

Example—the Euclidean spaces

Example—spheres

Example—the Lorentz-Minkowski space

Example—hyperbolic spaces

Exercise 1-1

Problem (Ex. 1-1)

Let $U \subset \mathbb{R}^n$ be a domain and g a Riemannian metric on U. Show that

1. *There exists an n-tuple of vector fields $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ such that*

$$g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta - ij = \begin{cases} 1 & (i = j) \\ 0 & (\textit{otherwise}) \end{cases}.$$

2. *Take another n-tuple $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ satisfying $g(\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{ij}$. Then there exists a matrix-valued function*

$$\Theta \colon U \to \mathrm{O}(n) \qquad [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta.$$

Exercise 1-2

Problem (Ex. 1-2)

Let $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$ *be the Minkowski vector space. Show that if* $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ *satisfies* $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = -1$*, the orthogonal complement*

$$\boldsymbol{v}^{\perp} := \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \,;\, \langle \boldsymbol{v}, \boldsymbol{x} \rangle_L = 0\}$$

is an n*-dimensional space-like subspace of* $\mathbb{R}^{n+1}_1$*.*