Advanced Topics in Geometry F (MTH.B502)

Riemannian connection

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Exercise 1-1

Problem

Let $U \subset \mathbb{R}^n$ be a domain and g a Riemannian metric on U. Show that

1. *There exists an n-tuple of vector fields $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ such that*

$$g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (\textit{otherwise}) \end{cases}.$$

orthonormal basis

2. *Take another n-tuple $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ satisfying $g(\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{ij}$. Then there exists a matrix-valued function*

$$\Theta \colon U \to \mathrm{O}(n) \qquad [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta.$$

1. Apply Gram-Schmidt orthogonalization for

$\exists$ basis (frame) $\underline{\left\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n} \right\}}$ may not exist globally $\left(= \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots \right\} \right)$

$\left(g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) : U \xrightarrow[C^\infty]{} \mathbb{R} \right)$

$\leadsto \exists \{ e_1, \cdots e_n \}$ orthonormal frame (exists locally)

2. $[e_1 \cdots e_n]$, $[v_1 \cdots v_n]$ orthonormal frames.

$\Rightarrow \underbrace{[e_1 \cdots e_n]}_{\text{orthonormal}} = [v_1 \cdots v_n] \exists \Theta$ a gauge transf.

$\Theta : U \xrightarrow{C^\infty} O(n)$

$e_j = \sum \Theta_j^k v_k$ $\qquad$ C^∞ $\qquad \Theta_j^\ell = g(e_j, v_\ell) : C^\infty$

C^∞

Exercise 1-2

$X = \begin{pmatrix} X^0 \\ X^1 \\ \vdots \\ X^n \end{pmatrix}$ $\langle X, X \rangle_L = -(X^0)^2 + (X^1)^2 + \cdots + (X^n)^2$

Problem

Let $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$ *be the Minkowski vector space. Show that if* $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ *satisfies* $\langle \boldsymbol{v}, \boldsymbol{v} \rangle_L = -1$*, the orthogonal complement* 直交補空間

e.g. $v = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\boldsymbol{v}^\perp := \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \,;\, \langle \boldsymbol{v}, \boldsymbol{x} \rangle_L = 0\} \ni x$$

is an n*-dimensional space-like subspace of* $\mathbb{R}^{n+1}_1$*.*

$\langle x, x \rangle > 0$ whenever $x \neq 0$.

$v^\perp = \mathrm{Ker}\{x \longmapsto \underbrace{\langle v, x \rangle}_{\in \mathbb{R}}\}$

$v^\flat$

$\dim \mathrm{Im}\, v^\flat = 1 \quad (\because \mathrm{Im}\, v^\flat \neq \{0\})$

$\dim \mathrm{Ker}\, v^\flat = \dim \mathbb{R}^{n+1}_1 - \dim \mathrm{Im}\, v^\flat = n$

$$v = \begin{pmatrix} v^0 \\ v^1 \\ \vdots \\ v^n \end{pmatrix} = v^0 e_0 + \vec{v} \qquad e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 0 \\ v^1 \\ \vdots \\ v^n \end{pmatrix}$$

$$\vec{v} \in \mathbb{R}^n \subset \mathbb{R}^{n+1}_1$$

$$\langle v, v \rangle_L = -1 \iff -(v^0)^2 + |\vec{v}|^2 = -1$$

$$\langle x, v \rangle_L = 0 \iff -x^0 v^0 + \langle \vec{v}, \vec{x} \rangle = 0$$

$$\langle x, x \rangle_L = -(x^0)^2 + \langle \vec{x}, \vec{x} \rangle$$

Cauchy-Schwarz für $\mathbb{R}^n$:

$$= -\frac{\langle \vec{v}, \vec{x} \rangle^2}{(v^0)^2} + \langle \vec{x}, \vec{x} \rangle$$

$$= \frac{1}{(v^0)^2} \left(\langle \vec{x}, \vec{x} \rangle (v^0)^2 - \langle \vec{v}, \vec{x} \rangle^2 \right)$$

$$= \frac{1}{(v^0)^2} \left(\langle \vec{x}, \vec{x} \rangle \{ \langle \vec{v}, \vec{v} \rangle \cancel{+1} \} - \langle \vec{v}, \vec{x} \rangle^2 \right) \geq 0$$

smooth fct.

the set of vector fields.

$\mathcal{F}(M)$-module

✓ $X,\ Y \in \mathfrak{X}(M)$

Xf

$$[X, Y]\colon \mathcal{F}(M) \ni f \longmapsto X(Yf) - Y(Xf) \in \mathcal{F}(M).$$

$\mathfrak{X}(M)$

$$[X, Y](fg) = f[X, Y]g + g[X, Y]f$$

$\frac{\partial}{\partial x_i} f$

- bilinear, skew-symmetric
- $[fX, Y] = f[X, Y] - (Yf)X$
- $[X, fY] = f[X, Y] + (Xf)Y$
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = \mathbf{0}$,

Jacobi's identity

Xf [illegible]

fX : [illegible]

Lie Bracket as integrability

Fact (Fact 2.1)

lin indep.

Let $(X_1, \dots, X_n)$ be an n-tuple of vector fields on n-dimensional manifolds, which is linearly independent in T_pM for each $p \in M$. Then existence of local coordinate system $(x^1, \dots, x^n)$ around p such that $\partial/\partial x^j = X_j$ $(j = 1, \dots, n)$ is equivalent to that $[X_j, X_k] = \mathbf{0}$ holds for all j, $k = 1, \dots, n$.

$\exists (x^j)$ $\quad \boxed{\dfrac{\partial}{\partial x^j} = X_j} \quad \Leftrightarrow \quad [X_j, X_k] = 0$

$\Rightarrow$

$$\left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right] f = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k} f - \frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j} f$$

$= 0$

$\boxed{\omega_p \in T_p^* M}$

(covariant 1-tensor
(0.1)-tensor)
1阶共变张量
1次微分形式

Lemma $\boxed{\text{(Lemma 2.2)}}$

A linear map $\boxed{\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)}$ *is a* 1*-form if and only if*

$$\omega(fX) = f\omega(X) \qquad (f \in \mathcal{F}(M), X \in \mathfrak{X}(M)).$$

$\Rightarrow \exists\, \omega_p \in T_p^* M$ for $\forall p$

Lemma (Lemma 2.3)

A bilinear map $\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$ *is a* $(0,2)$*-tensor if and only if*

$$(*)\ \alpha(fX, Y) = \alpha(X, fY) = f\alpha(X, Y) \qquad (f \in \mathcal{F}(M), X, Y \in \mathfrak{X}(M))$$

$f(x)$

holds.

$\langle [X, Y], v \rangle$: does not satisfy $(*)$

$\alpha(X, Y)$

$\Gamma(T^*M \otimes T^*M)$

- $\omega : M \xrightarrow{C^\infty} T^*M \qquad \omega_p \in T_p^*M$

  $\omega_p : T_pM \to \mathbb{R}$ linear.

- $\omega(X)(p) = \omega_p(X_p) \in \mathbb{R} \qquad X \in \mathfrak{X}(M)$

  $\rightsquigarrow \underline{\omega(X) \in \mathcal{F}(M)}$

1-form $\omega \quad \longmapsto \quad \omega : \mathfrak{X}(M) \to \mathcal{F}(M)$

No in general

✓ additional condition : $\underline{\omega : \mathcal{F}(M)\text{-linear}}$

$$\omega\left(\sum X^j \frac{\partial}{\partial x^j}\right) = \underline{\sum X^j \, \omega\left(\frac{\partial}{\partial x^j}\right)}$$

pointwise

Differential Forms

- $\wedge^0(M) := \mathcal{F}(M)$, 0- form.
- $\wedge^1(M) := \Gamma(T^*M)$, 1- form
- $\wedge^2(M) := \{\omega \in \Gamma(T^*M \otimes T^*M); \text{skew-symmetric})\}$

2 - form

$$\boxed{\alpha(X, Y) = -\alpha(Y, X)}$$

Exterior product 外積

$\Gamma(T^*M) = \Lambda^1(M)$ $\qquad\qquad$ $\Lambda^2(M)$

$\qquad\quad \cup \qquad\qquad\qquad\qquad\qquad \cup$

$\qquad\quad \alpha, \beta \quad \longmapsto \quad (\alpha \wedge \beta)(X, Y)$

$\alpha(X) \quad \beta(Y) \qquad\qquad := \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$

- $f \in \wedge^0(M) \Rightarrow df(X) = Xf.$ $\underline{df \in \Lambda^1(M)}$
- $\underline{\alpha \in \wedge^1(M)} \Rightarrow d\alpha(\underbrace{X,Y}_{\text{vector fields}}) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$

$\boxed{d\alpha \in \Lambda^2(M)}$

$$d\alpha(fX, Y) = fX\alpha(Y) - Y\alpha(fX) - \alpha([fX, Y])$$

$$= fX\alpha(Y) - Y(f\alpha(X)) - \alpha(f[X, Y] - (Yf)X)$$

$$= fX\alpha(Y) - \cancel{(Yf)\alpha(X)} - fY\alpha(X) - f\alpha([X, Y]) + \cancel{Yf\alpha(X)}$$

$$= f\,d\alpha(X, Y) \quad \cdots$$

$$d\alpha(X, Y) = -d\alpha(Y, X)$$