Advanced Topics in Geometry F (MTH.B502)

Riemannian connection

Kotaro Yamada

kotaro@math.titech.ac.jp

http://www.math.titech.ac.jp/~kotaro/class/2022/geom-e/

Tokyo Institute of Technology

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Exercise 1-1

Problem

Let $U \subset \mathbb{R}^n$ be a domain and g a Riemannian metric on U. Show that

1. *There exists an n-tuple of vector fields $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ such that*

$$g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (\textit{otherwise}) \end{cases}.$$

2. *Take another n-tuple $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ satisfying $g(\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{ij}$. Then there exists a matrix-valued function*

$$\Theta \colon U \to \mathrm{O}(n) \qquad [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta.$$

Exercise 1-2

Problem

Let $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$ *be the Minkowski vector space. Show that if* $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ *satisfies* $\langle \boldsymbol{v}, \boldsymbol{v} \rangle_L = -1$*, the orthogonal complement*

$$\boldsymbol{v}^{\perp} := \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \,;\, \langle \boldsymbol{v}, \boldsymbol{x} \rangle_L = 0\}$$

is an n*-dimensional space-like subspace of* $\mathbb{R}^{n+1}_1$*.*

Lie Bracket

X, $Y \in \mathfrak{X}(M)$:

$$[X, Y]\colon \mathcal{F}(M) \ni f \longmapsto X(Yf) - Y(Xf) \in \mathcal{F}(M).$$

- bilinear, skew-symmetric
- $[fX, Y] = f[X, Y] - (Yf)X$
- $[X, fY] = f[X, Y] + (Xf)Y$
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = \mathbf{0}$,

Lie Bracket as integrability

Fact (Fact 2.1)

Let $(X_1, \ldots, X_n)$ *be an* n*-tuple of vector fields on* n*-dimensional manifolds, which is linearly independent in* T_pM *for each* $p \in M$*. Then existence of local coordinate system* $(x^1, \ldots, x^n)$ *around* p *such that* $\partial/\partial x^j = X_j$ $(j = 1, \ldots, n)$ *is equivalent to that* $[X_j, X_k] = \mathbf{0}$ *holds for all* j*,* $k = 1, \ldots, n$*.*

Tensors

Lemma (Lemma 2.2)

A linear map $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ *is a* 1*-form if and only if*

$$\omega(fX) = f\omega(X) \qquad (f \in \mathcal{F}(M), X \in \mathfrak{X}(M)).$$

Lemma (Lemma 2.3)

A bilinear map $\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$ *is a* $(0,2)$*-tensor if and only if*

$$\alpha(fX, Y) = \alpha(X, fY) = f\alpha(X, Y) \qquad (f \in \mathcal{F}(M), X, Y \in \mathfrak{X}(M))$$

holds.

Differential Forms

- $\wedge^0(M) := \mathcal{F}(M)$,
- $\wedge^1(M) := \Gamma(T^*M)$,
- $\wedge^2(M) := \{\omega \in \Gamma(T^*M \otimes T^*M); \text{skew-symmetric})\}$

Exterior product

Exterior derivative

- $f \in \wedge^0(M) \Rightarrow df(X) = Xf$.
- $\alpha \in \wedge^1(M) \Rightarrow d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$