Advanced Topics in Geometry F (MTH.B502)

Riemannian connection

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Riemannian connections

"リーマン接続"

Lemma (Lemma 2.7)

(Differentiation of vector fields)

There exists the unique bilinear map
$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{X}(M)$ *satisfying*

∇

"nabla"

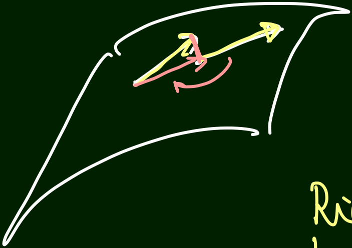
"atled"

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

Y

$$X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle \cancel{X}, \nabla_{\cancel{Y}} Z\rangle$$

X



Xf

$\langle Y, Z\rangle = g(Y, Z) \in \mathcal{F}(M)$

Riemannian connection

Levi-Civita 〃

Linear Connections

Lemma (Lemma 2.9)

The Riemannian connection ∇ satisfies

$$\nabla_{fX}Y = f\nabla_X Y, \qquad \nabla_X(fY) = (Xf)Y + f\nabla_X Y. \quad ⊛$$

In general:

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \xrightarrow{\text{bilinear}} \mathfrak{X}(M) \text{ with } ⊛ :$$

a linear connection, an affine connection on TM

線形接続

Definition

Let $U \subset M$ be a domain of M. An n-tuple of vector fields $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ on U is called an orthonormal frame on U. if $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij}$.
It is said to be positive if M is oriented and $\{\boldsymbol{e}_j\}$ is compatible to the orientation on M.

(与えた順)

- $\{\omega^1 \cdots \omega^n\}$ the dual frame

$$\omega^j(\boldsymbol{e}_k) = \delta^j_k = \begin{cases} 1 & j = k \\ 0 & \text{otherwise} \end{cases}$$

$$\omega^j \in \Lambda^1(M)$$

$$\omega^j = \langle \boldsymbol{e}_j, * \rangle$$

Gauge Transformations

Lemma (Lemma 2.13)

Let $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *be two orthonormal frames on* $U \subset M$*. Then there exists a smooth map*

gauge transf.

$$\Theta\colon U \longrightarrow \mathrm{O}(n) \qquad \textit{such that} \qquad [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta.$$

Moreover, if $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *determines the common orientation,* Θ *is valued on* $\mathrm{SO}(n)$*.*

Connection Forms

Definition (Definition 2.15)

The connection form with respect to an orthonormal frame $\{\boldsymbol{e}_j\}$ is a $n \times n$-matrix valued one form Ω on U defined by

$$\Omega = \begin{pmatrix} \omega_1^1 & \omega_2^1 & \dots & \omega_n^1 \\ \omega_1^2 & \omega_2^2 & \dots & \omega_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^n & \omega_2^n & \dots & \omega_n^n \end{pmatrix}, \qquad \omega_j^k := \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle \in \wedge^1(U).$$

matrix-valued 1-form

$({}^t\Omega = -\Omega)$

$$\nabla \boldsymbol{e}_j = \sum_{k=1}^{n} \omega_j^k \boldsymbol{e}_k$$

$$\nabla[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Omega$$

$\{e_1 \cdots e_n\}$: orthonormal frame.

$$\nabla e_j : \mathfrak{X}(M) \ni X \longmapsto \nabla_X e_j \in \mathfrak{X}(M)$$

$$\boxed{\nabla_{fX} Y = f \nabla_X Y}$$

$$\nabla_X e_j = \sum_{k=1}^{n} \omega_j^k(X) e_k$$

$\mathfrak{X}(M) \ni X \longmapsto \omega_j^k(X) \in \mathfrak{F}(M)$ determines a 1-form

$$\because \ \omega_j^k(fX) = f\,\omega_j^k(X)$$

$$\omega_j^k := \langle \nabla e_j, e_k \rangle = \omega^k(\nabla e_j)$$

Exercise 2-1

Problem (Ex. 2-1)

Let $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ be two orthonormal frames on a domain U of a Riemannian n-manifold M, which are related as (~~??~~). Show that the connection forms Ω of $\{\boldsymbol{e}_j\}$ and Λ of $\{\boldsymbol{v}_j\}$ satisfy
$\boxed{\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta.}$

Exercise 2-2

Problem (Ex. 2-2)

Let $\mathbb{R}^3_1$ *be the* 3*-dimensional Lorentz-Minkowski space and let* $H^2(-c^2)$ *the hyperbolic* 2*-space (i.e. the hyperbolic plane) as defined in Example* **??**. *Verify that*

$$\boxed{(u, v) \mapsto \left(\frac{1}{c}\cosh cu, \frac{\cos v}{c}\sinh cu, \frac{\sin v}{c}\sinh cu\right)}$$

gives a local coordinate system on $U := H^2(-c^2) \setminus \{(1/c, 0, 0)\}$, *and*

$$\boldsymbol{e}_1 := (\sinh cu, \cos v \cosh cu, \sin v \cosh cu), \qquad \boldsymbol{e}_2 := (0, -\sin v, \cos v)$$

forms a orthonormal frame on U.

$$\left\{ (x^0, x^1, x^2) \;\middle|\; -(x^0)^2 + (x^1)^2 + (x^2)^2 = \frac{-1}{c^2}, \; x^0 > 0 \right\} \subset \mathbb{R}^3_1$$