Advanced Topics in Geometry F (MTH.B502)

Riemannian connection

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Riemannian connections

Lemma (Lemma 2.7)

There exists the unique bilinear map
$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{X}(M)$ *satisfying*

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= [X, Y], \\ X \langle Y, Z \rangle &= \langle \nabla_X Y, Z \rangle + \langle X, \nabla_Y Z \rangle \end{aligned}$$

Linear Connections

Lemma (Lemma 2.9)

The Riemannian connection ∇ satisfies

$$\nabla_{fX}Y = f\nabla_X Y, \qquad \nabla_X(fY) = (Xf)Y + f\nabla_X Y.$$

Orthonormal Frame

Definition

Let $U \subset M$ be a domain of M. An n-tuple of vector fields $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ on U is called an orthonormal frame on U. if $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij}$.

It is said to be positive if M is oriented and $\{\boldsymbol{e}_j\}$ is compatible to the orientation on M.

Gauge Transformations

Lemma (Lemma 2.13)

Let $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *be two orthonormal frames on* $U \subset M$*. Then there exists a smooth map*

$$\Theta \colon U \longrightarrow \mathrm{O}(n) \qquad \text{such that} \qquad [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta.$$

Moreover, if $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *determines the common orientation,* Θ *is valued on* $\mathrm{SO}(n)$*.*

Connection Forms

Definition (Definition 2.15)

The connection form with respect to an orthonormal frame $\{\boldsymbol{e}_j\}$ is a $n \times n$-matrix valued one form Ω on U defined by

$$\Omega = \begin{pmatrix} \omega_1^1 & \omega_2^1 & \dots & \omega_n^1 \\ \omega_1^2 & \omega_2^2 & \dots & \omega_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^n & \omega_2^n & \dots & \omega_n^n \end{pmatrix},$$

$$\omega_j^k := \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle \in \wedge^1(U).$$

$$\nabla \boldsymbol{e}_j = \sum_{k=1}^{n} \omega_j^k \boldsymbol{e}_k$$

$$\nabla[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Omega$$

Exercise 2-1

Problem (Ex. 2-1)

Let $\{e_j\}$ *and* $\{v_j\}$ *be two orthonormal frames on a domain* U *of a Riemannian* n*-manifold* M*, which are related as* (**??**)*. Show that the connection forms* Ω *of* $\{e_j\}$ *and* Λ *of* $\{v_j\}$ *satisfy* $\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta$*.*

Exercise 2-2

Problem (Ex. 2-2)

*Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-c^2)$ the hyperbolic 2-space (i.e. the hyperbolic plane) as defined in Example **??**. Verify that*

$$(u,v) \mapsto \left(\frac{1}{c}\cosh cu, \frac{\cos v}{c}\sinh cu, \frac{\sin v}{c}\sinh cu\right)$$

gives a local coordinate system on $U := H^2(-c^2) \setminus \{(1/c, 0, 0)\}$, and

$$\boldsymbol{e}_1 := (\sinh cu, \cos v \cosh cu, \sin v \cosh cu), \qquad \boldsymbol{e}_2 := (0, -\sin v, \cos v)$$

forms a orthonormal frame on U.