Advanced Topics in Geometry F (MTH.B502)

Curvature as integrability

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-f/

Tokyo Institute of Technology

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Addendum 1

Proposition (Prop 3.1; The local expression of the Lie bracket)

Let $(U; x^1, \ldots, x^n)$ be a coordinate neighborhood of an n-manifold M. Then the Lie bracket of two vector fields

$$X = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j}, \qquad Y = \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}$$

is expressed as

$$[X, Y] = \sum_{j=1}^{n} \left(\xi^k \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \xi^j}{\partial x^k} \right) \frac{\partial}{\partial x^j}.$$

$$[X, Y]f = X(Yf) - Y(Xf)$$

- $$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] f = \frac{\partial}{\partial x^i}\frac{\partial f}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial f}{\partial x^i} = 0$$

- $$[X, hY] = h[X, Y] + (Xh)Y$$

$$[hX, Y] = h[X, Y] - (Yh)X.$$

- $$\left[\sum_j \xi^j \frac{\partial}{\partial x^j} \quad \sum_k \eta^k \frac{\partial}{\partial x^k}\right]$$

$$= \sum_{k,l}\left(\xi^l \frac{\partial \eta^k}{\partial x^l} - \eta^l \frac{\partial \xi^k}{\partial x^l}\right)\frac{\partial}{\partial x^k}$$

Addendum 2

Proposition (Prop 3.2)

Let U be a domain of a Riemannian n-manifold (M, g) and $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]$ an orthornomal frame on U. Then the connection form ω_i^j with respect to the frame $[\boldsymbol{e}_j]$ is obtained as

$$\omega_i^j(\boldsymbol{e}_k) = \frac{1}{2}\Big(\langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k \rangle - \langle [\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i \rangle + \langle [\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j \rangle \Big),$$

where $\langle\ ,\ \rangle$ denotes the inner product induced from g.

$$\nabla_X e_j = \sum_k \omega_j^k(X) e_k$$

$$\omega_i^j(e_k) = \langle \nabla_{e_k} e_i , e_j \rangle = e_k \cancel{\underbrace{\langle e_i\ e_j \rangle}_{\delta_{ij}}} - \langle e_i\ \nabla_{e_k} e_j \rangle$$

$$= - \langle e_i , \nabla_{e_k} e_j \rangle$$

$$= - \langle e_i\ \nabla_{e_j} e_k \rangle - \langle e_i\ [e_j\ e_k] \rangle$$

$$= - \cancel{e_j \langle e_i\ e_k \rangle} + \langle \nabla_{e_j} e_i\ e_k \rangle - \langle e_i , [e_j, e_k] \rangle$$

$$= \langle \nabla_{e_i} e_j , e_k \rangle + \langle [e_j, e_i], e_k \rangle - \langle e_i [e_j, e_k] \rangle$$

$$= - \langle e_j , \nabla_{e_i} e_k \rangle + \sim\sim\sim$$

$$= \underbrace{- \langle e_j\ \nabla_{e_k} e_i \rangle}_{-\omega_i^j(e_k)} - \underline{\langle [e_i\ e_k], e_j \rangle + \sim}$$

$$\nabla_X Y - \nabla_Y X = [X\ Y]$$

$$X \langle Y. Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y\ \nabla_X Z \rangle$$

Exercise 2-1

$\nabla e_j = \sum \omega_j^k e_k$

Problem (Ex. 2-1)

Let $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ be two orthonormal frames on a domain U of a Riemannian n-manifold M, which are related as (2.11). Show that the connection forms Ω of $\{\boldsymbol{e}_j\}$ and Λ of $\{\boldsymbol{v}_j\}$ satisfy
$\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta.$

$U \to SO(n)$

gauge transf.

$$[e_1 \ \cdots \ e_n] = [v_1 \ \cdots \ v_n]\,\Theta$$

$$\nabla [e_1 \ \cdots \ e_n] = [e_1 \ \cdots \ e_n]\begin{pmatrix} \omega_1^1 & \cdots & \omega_n^1 \\ \omega_1^i & \ddots & \\ \vdots & & \ddots \\ \omega_1^n & & \omega_n^n \end{pmatrix}$$

$$\nabla [e_1 \cdots e_n] = [e_1 \cdots e_n]\,\Omega \qquad \Omega = (\omega^j_i)$$

row

$$\nabla [v_1 \cdots v_n] = \underline{[v_1 \cdots v_n]\,\Lambda}$$

$$\nabla \left([v_1 \cdots v_n]\,\Theta \right)$$

function

$$\nabla_X f v = \underbrace{(df)}_{(Xf)} v + \underline{f \nabla_X v}$$

$$= \underline{(\nabla [v_1 \cdots v_n])\,\Theta} + \underline{[v_1 \cdots v_n]\, d\Theta}$$

$$= [v_1 \cdots v_n] \left(\Lambda \Theta + d\Theta \right)$$

$$= [e_1 \cdots e_n]\, \Theta^{-1} \left(\Lambda \Theta + d\Theta \right)$$

Exercise 2-2

Problem (Ex. 2-2)

Let $\mathbb{R}^3_1$ *be the* 3*-dimensional Lorentz-Minkowski space and let* $H^2(-c^2)$ *the hyperbolic* 2*-space (i.e. the hyperbolic plane) as defined in Example* **??**. *Verify that*

$$(u, v) \mapsto \left(\frac{1}{c}\cosh cu, \frac{\cos v}{c}\sinh cu, \frac{\sin v}{c}\sinh cu\right)$$

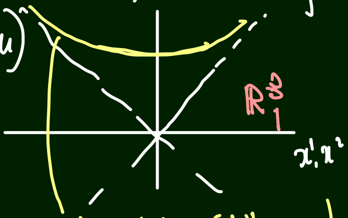
gives a local coordinate system on $U := H^2(-c^2) \setminus \{(1/c, 0, 0)\}$*, and*

$$\begin{aligned} \boldsymbol{e}_1 &:= (\sinh cu, \cos v \cosh cu, \sin v \cosh cu), \\ \boldsymbol{e}_2 &:= (0, -\sin v, \cos v) \end{aligned}$$

forms a orthonormal frame on U*.*

$p : (u,v) \mapsto \left(\frac{1}{c}\cosh cu, \frac{\cos v}{c}\sinh cu, \frac{\sin v}{c}\sinh cu\right) \in \{(x^0, x^1, x^2) ; -(x^0)^2 + (x^1)^2 + (x^2)^2 = -\frac{1}{c^2}, \; x^0 > 0\}$

$(0,\infty) \times S^1$



e_1

$p_u = (\sinh cu, \cos v \cosh cu, \sin v \sinh cu)$

$\langle p_u, p_u \rangle = -\sinh^2 cu + \cosh^2 cu = 1$

$p_v = \left(0, -\frac{1}{c}\sin v, \frac{1}{c}\cos v\right)\cdot \sinh cu \qquad -x_0^2 + (x^1)^2 + (x^2)^2 = -\frac{1}{c^2}$

$\langle \; , \; \rangle : (- \; + \; +)$

$e_2 = \frac{c}{\sinh cu} p_v = (0, -\sin v, \cos v)$

$\langle e_1, e_2 \rangle = 0, \quad |e_j|^2 = 1 \qquad : \quad \{e_1, e_2\} :$ orthonormal

$\left(\frac{c}{\sinh cu}\frac{\partial}{\partial v}\right)$

$\cdot \; e_1 = dp\left(\frac{\partial}{\partial u}\right) = p_u \left(\frac{\partial}{\partial u}\right) \qquad e_2 = \frac{c}{\sinh cu} dp\left(\frac{\partial}{\partial v}\right)$