Advanced Topics in Geometry F (MTH.B502)

Curvature as integrability

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Addendum 1

Proposition (Prop 3.1; The local expression of the Lie bracket)

Let $(U; x^1, \ldots, x^n)$ *be a coordinate neighborhood of an* n*-manifold* M*. Then the Lie bracket of two vector fields*

$$X = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j}, \qquad Y = \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}$$

is expressed as

$$[X, Y] = \sum_{j=1}^{n} \left(\xi^k \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \xi^j}{\partial x^k} \right) \frac{\partial}{\partial x^j}.$$

Addendum 2

Proposition (Prop 3.2)

Let U be a domain of a Riemannian n-manifold (M, g) and $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]$ an orthornomal frame on U. Then the connection form ω_i^j with respect to the frame $[\boldsymbol{e}_j]$ is obtained as

$$\omega_i^j(\boldsymbol{e}_k) = \frac{1}{2}\Big(\langle[\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k\rangle - \langle[\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i\rangle + \langle[\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j\rangle\Big),$$

where $\langle\ ,\ \rangle$ denotes the inner product induced from g.

Exercise 2-1

Problem (Ex. 2-1)

Let $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *be two orthonormal frames on a domain* U *of a Riemannian* n*-manifold* M*, which are related as (2.11). Show that the connection forms* Ω *of* $\{\boldsymbol{e}_j\}$ *and* Λ *of* $\{\boldsymbol{v}_j\}$ *satisfy* $\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta$.

Exercise 2-2

Problem (Ex. 2-2)

Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-c^2)$ the hyperbolic 2-space (i.e. the hyperbolic plane) as defined in Example **??**. *Verify that*

$$(u, v) \mapsto \left(\frac{1}{c} \cosh cu, \frac{\cos v}{c} \sinh cu, \frac{\sin v}{c} \sinh cu \right)$$

gives a local coordinate system on $U := H^2(-c^2) \setminus \{(1/c, 0, 0)\}$, and

$$\begin{aligned} \boldsymbol{e}_1 &:= (\sinh cu, \cos v \cosh cu, \sin v \cosh cu), \\ \boldsymbol{e}_2 &:= (0, -\sin v, \cos v) \end{aligned}$$

forms a orthonormal frame on U.