Advanced Topics in Geometry F (MTH.B502)

Curvature as integrability

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-f/

Tokyo Institute of Technology

2022/07/05

The Integrability Condition

Theorem (Theorem 3.7)

Let Ω be an $\mathrm{M}_n(\mathbb{R})$-valued 1-form on a simply connected m-manifold M satisfying

$$\boxed{d\Omega + \Omega \wedge \Omega = O.}$$

Then for each $\mathrm{P}_0 \in M$ and $F_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $F\colon M \to \mathrm{M}_n(\mathbb{R})$ satisfying (??) (*) with $F(\mathrm{P}) = F_0$. Moreover,

- *if $F_0 \in \mathrm{GL}(n, \mathbb{R})$, $F(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ holds on M,*
- *if $F_0 \in \mathrm{SO}(n)$ and Ω is skew-symmetric, $F(\mathrm{P}) \in \mathrm{SO}(n)$ holds on M.*

(*) $\boxed{dF = F\Omega}$ $\qquad F(P_0) = F_0$

$\mathbb{R}^m \supset U$: a domain $\quad \Omega_l : U \to M_n(\mathbb{R}) \quad l = 1 \cdots m$

$n \times n$ matrices $/ \mathbb{R}$

$$(*) \quad \frac{\partial F}{\partial x^l} = F \Omega_l \quad (l = 1, \cdots, m)$$

$F(P_0) = F_0 \in GL(n, \mathbb{R})$ initial condition

$\exists ?\ F : U \to GL(n, \mathbb{R})$ satisfying $(*)$

$$\frac{\partial^2 F}{\partial x^k \partial x^l} = \frac{\partial^2 F}{\partial x^l \partial x^k}$$

nec. condition

$$(\heartsuit) \quad \frac{\partial \Omega_l}{\partial x^k} - \frac{\partial \Omega_k}{\partial x^l} = \Omega_k \Omega_l - \Omega_l \Omega_k$$

integrability condition

$\boxed{\text{Thm}}$ If U : simply conn, $(\heartsuit)$: sufficient cond.

$$(*)\quad \frac{\partial F}{\partial x^\ell} = F\,\Omega_\ell \qquad (\ell = 1, \cdots, m).$$

$$dF = \sum_\ell \frac{\partial F}{\partial x^\ell}\, dx^\ell$$

coordinate free

$$\Omega := \sum_\ell \Omega_\ell\, dx^\ell$$

matrix-valued 1-form

$$\boxed{(*)\quad dF = F\,\Omega}$$

make sense on manifolds.

function, 1 form

$$d(f\omega) = df \wedge \omega + f\,d\omega$$

Integrability condition.

$$0 = d(dF) = d(F\Omega) =$$

$$= \underline{dF \wedge \Omega} + F\,d\Omega = F\left(\boxed{\Omega\wedge\Omega + d\Omega}\right)$$

matrix multiplication with $\wedge$

Poincaré lemma

Theorem (Theorem 3.8)

If a differential 1-form ω defined on a simply connected and connected m-manifold M is closed, that is, $d\omega = 0$ holds, then there exists a C^∞-function f on U such that $df = \omega$. Such a function f is unique up to additive constants.

Curvature Form

- (M, g): a Riemannian n-manifold.
- $U \subset M$: a domain
- $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$: an orthonormal frame.
- $\Omega = (\omega_i^j)$: the connection form with respect to $[\boldsymbol{e}_j]$.

$n \times n$-matrix-valued 1-form skew-symmetric.

Definition (Definition 3.9)

$K := d\Omega + \Omega \wedge \Omega$: the curvature form 曲率形式 w.r.to $[\boldsymbol{e}_j]$

$$\Omega \wedge \Omega = \left(\sum_k \left(\omega_k^j \wedge \omega_i^k \right) \right)_{i,j=1 \dots n}$$

Gauge Transformations

- $\Theta\colon U \to \mathrm{SO}(n)$: $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta$ ← gauge transf.
- $\widetilde{\Omega}$: the connection form w. r. to $[\boldsymbol{v}_j]$
- $\widetilde{K}$: the curvature form w. r. to $[\boldsymbol{v}_j]$

Proposition (Prop. 3.10)

1. $\Omega = \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta$, ← Exercise 2-1
2. $K = \Theta^{-1}\widetilde{K}\Theta$.

Rem "$K = O$" does not depend on frames.

$\Leftrightarrow$ (M, g) is flat
平坦

$$K = d\Omega + \Omega \wedge \Omega \qquad \Omega = \Theta^{-1}\hat{\Omega}\Theta + \Theta^{-1}d\Theta$$

$$= d\left(\Theta^{-1}\hat{\Omega}\Theta + \Theta^{-1}d\Theta\right) + \left(\Theta^{-1}\hat{\Omega}\Theta + \Theta^{-1}d\Theta\right) \wedge \left(\text{———} \right)$$

$$= \left(d\Theta^{-1}\right)\wedge\hat{\Omega}\Theta + \Theta^{-1}d\hat{\Omega}\Theta \; \boxed{-} \; \Theta^{-1}\hat{\Omega}\wedge d\Theta$$

$$d\Theta^{-1} = -\Theta^{-1}d\Theta\,\Theta^{-1}$$

$$+ \, d\Theta^{-1}\wedge d\Theta + \cancel{\Theta^{-1}dd\Theta}$$

$$+ \left(\Theta^{-1}\hat{\Omega}\Theta + \Theta^{-1}d\Theta\right)\wedge\left(\text{———}\right)$$

$$= -\Theta^{-1}d\Theta\,\Theta^{-1}\wedge\hat{\Omega}\Theta + \cdots \quad \cdot -$$

function

$$= \Theta^{-1}\left(d\hat{\Omega} + \hat{\Omega}\wedge\hat{\Omega}\right)\Theta \; = \; \Theta^{-1}\hat{K}\Theta$$

Flatness

Theorem (Theorem 3.11)

Let U be a domain of a Riemannian n-manifold (M, g) and K the curvature form with respect to an orthonormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ on U. For a point $\mathrm{P} \in U$, there exists a local coordinate system $(x^1, \dots, x^n)$ around P such that $[\partial/\partial x^1, \dots, \partial/\partial x^n]$ is an orthonormal frame if and only if K vanishes on a neighborhood of P.

$\exists$ local coordinates $(x^1, \dots x^n)$ s.t

$\left\{\frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^n}\right\}$: orthonormal

$\Leftrightarrow$ $K = 0$ (flat)

• Assume $\left[\frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^n}\right] = [e_1 \cdots e_n]$: orthonormal

$\Rightarrow \underline{\omega_i^j} = 0 \quad \left(\because \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right] = [e_j, e_k] = 0\right)$

↑ the connection form w.r.t. $[e_j]$

$\Rightarrow K = d\Omega + \Omega \wedge \Omega = 0$: flat.

- Assume $K = d\Omega + \Omega\wedge\Omega$ w.r.to $[e_1 \cdots e_n]$
$= 0$ (Ω is not necessarily 0)

- Find an orthonormal frame $[v_1 \cdots v_n]$ s.t. $\boxed{\tilde{\Omega} = 0}$

$$[e_1 \cdots e_n] = [v_1 \cdots v_n]\,\Theta$$

$$\Omega = \Theta^{-1}\tilde{\Omega}\,\Theta + \Theta^{-1}d\Theta$$

$$\Theta^{-1}\tilde{\Omega}\,\Theta = 0$$

$\Leftrightarrow$ Solve $\boxed{d\Theta = \Theta\,\Omega}$ integrability

$\exists\,\Theta$ because $d\Omega + \Omega\wedge\Omega$

$\hat{\tilde{\Omega}} = 0 \Rightarrow$ (to be continued)

$\Rightarrow d\hat{\tilde{\omega}}^i = 0$ where $\begin{pmatrix} \tilde{\omega}^1 \\ \vdots \\ \tilde{\omega}^n \end{pmatrix}$: dual to $[v_1 \cdots v_n]$

$\Rightarrow \exists x^i$ s.t. $dx^i = \tilde{\omega}^i$

desired coordinate system

Exercise 3-1

Problem (Ex. 3-1)

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \qquad \textit{on} \qquad U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r\to+0} \varphi(r) \to 0, \qquad \lim_{r\to+0} \varphi'(r) \to 1.$$

Find a function φ *such that* (U, g) *is flat.*
(Hint: $[\partial/\partial r, (1/\varphi)\partial/\partial\theta)]$ *is an orthonormal frame.)*

$\circledast_1$ $\circledast_2$

Exercise 3-2

Problem (Ex. 3-2)

Compute the curvature form of $H^2(-c^2)$ *with respect to an orthonormal frame* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *as in Exercise 2-2*

Exercise 2-2

Problem (Ex. 2-2)

Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-c^2)$ the hyperbolic 2-space (i.e. the hyperbolic plane) as defined in Example ??. Verify that

$$(u,v) \mapsto \left(\frac{1}{c}\cosh cu, \frac{\cos v}{c}\sinh cu, \frac{\sin v}{c}\sinh cu\right)$$

gives a local coordinate system on $U := H^2(-c^2) \setminus \{(1/c, 0, 0)\}$, and

$$\begin{aligned} \boldsymbol{e}_1 &:= (\sinh cu, \cos v \cosh cu, \sin v \cosh cu), \\ \boldsymbol{e}_2 &:= (0, -\sin v, \cos v) \end{aligned}$$

forms a orthonormal frame on U.