Advanced Topics in Geometry F (MTH.B502)

Curvature as integrability

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2022/07/05

The Integrability Condition

Theorem (Theorem 3.7)

Let Ω *be an* $\mathrm{M}_n(\mathbb{R})$*-valued* 1*-form on a simply connected* m*-manifold* M *satisfying*

$$d\Omega + \Omega \wedge \Omega = O.$$

Then for each $\mathrm{P}_0 \in M$ *and* $F_0 \in \mathrm{M}_n(\mathbb{R})$*, there exists the unique* $n \times n$*-matrix valued function* $F \colon M \to \mathrm{M}_n(\mathbb{R})$ *satisfying* (**??**) *with* $F(\mathrm{P}) = F_0$*. Moreover,*

- *if* $F_0 \in \mathrm{GL}(n, \mathbb{R})$*,* $F(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ *holds on* M*,*
- *if* $F_0 \in \mathrm{SO}(n)$ *and* Ω *is skew-symmetric,* $F(\mathrm{P}) \in \mathrm{SO}(n)$ *holds on* M*.*

Poincaré lemma

Theorem (Theorem 3.8)

If a differential 1*-form* ω *defined on a simply connected and connected* m*-manifold* M *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

Curvature Form

- (M, g): a Riemannian n-manifold.
- $U \subset M$: a domain
- $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$: an orthonormal frame.
- $\Omega = (\omega_i^j)$: the connection form with respect to $[\boldsymbol{e}_j]$.

Definition (Definition 3.9)

$K := d\Omega + \Omega \wedge \Omega$: the curvature form

Gauge Transformations

- $\Theta\colon U \to \mathrm{SO}(n)$: $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]\Theta$
- $\widetilde{\Omega}$: the connection form w. r. to $[\boldsymbol{v}_j]$
- $\widetilde{K}$: the curvature form w. r. to $[\boldsymbol{v}_j]$

Proposition (Prop. 3.10)

1. $\Omega = \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta$,
2. $K = \Theta^{-1}\widetilde{K}\Theta$.

Flatness

Theorem (Theorem 3.11)

Let U be a domain of a Riemannian n-manifold (M, g) and K the curvature form with respect to an orthonormal frame $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]$ on U. For a point $\mathrm{P} \in U$, there exists a local coordinate system $(x^1, \ldots, x^n)$ around P such that $[\partial/\partial x^1, \ldots, \partial/\partial x^n]$ is an orthonormal frame if and only if K vanishes on a neighborhood of P.

Exercise 3-1

Problem (Ex. 3-1)

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \qquad \text{on} \qquad U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r\to+0} \varphi(r) \to 0, \qquad \lim_{r\to+0} \varphi'(r) \to 1.$$

Find a function φ *such that* (U, g) *is flat.*
(Hint: $[\partial/\partial r, (1/\varphi)\partial/\partial\theta)]$ *is an orthonormal frame.)*

Exercise 3-2

Problem (Ex. 3-2)

Compute the curvature form of $H^2(-c^2)$ *with respect to an orthonormal frame* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *as in Exercise 2-2*

Exercise 2-2

Problem (Ex. 2-2)

*Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-c^2)$ the hyperbolic 2-space (i.e. the hyperbolic plane) as defined in Example **??**. Verify that*

$$(u,v) \mapsto \left(\frac{1}{c}\cosh cu, \frac{\cos v}{c}\sinh cu, \frac{\sin v}{c}\sinh cu\right)$$

gives a local coordinate system on $U := H^2(-c^2) \setminus \{(1/c, 0, 0)\}$, and

$$\begin{aligned} \boldsymbol{e}_1 &:= (\sinh cu, \cos v \cosh cu, \sin v \cosh cu), \\ \boldsymbol{e}_2 &:= (0, -\sin v, \cos v) \end{aligned}$$

forms a orthonormal frame on U.