Advanced Topics in Geometry F (MTH.B502)

SEctional Curvature

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Proposition (Prop 3.1; The local expression of the Lie bracket)

Let $(U; x^1, \dots, x^n)$ be a coordinate neighborhood of an n-manifold M. Then the Lie bracket of two vector fields

$$X = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j} \qquad Y = \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}$$

$$[X, Y]f = X(Yf) - Y(Xf)$$

is expressed as

$$[X, Y] = \sum_{j=1}^{n} \left(\xi^k \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \xi^j}{\partial x^k} \right) \frac{\partial}{\partial x^j}.$$

Addendum 2

$\left(\omega^k(e_j) = \delta^k_j\right)$

Proposition (Prop 3.2)

(ω^k) : dual basis

Let U be a domain of a Riemannian n-manifold (M, g) and $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ an orthornomal frame on U. Then the connection form ω_i^j with respect to the frame $[\boldsymbol{e}_j]$ is obtained as

$$\omega_i^j(\boldsymbol{e}_k) = \frac{1}{2}\Big(-\langle[\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k\rangle + \langle[\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i\rangle + \langle[\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j\rangle \Big),$$

where $\langle\ ,\ \rangle$ denotes the inner product induced from g.

$$\star\ \omega_i^j = \sum_{k=1}^{n} \omega_j^j(e_k)\, \omega^k$$

Exercise 3-1

Problem (Ex. 3-1)

"polar coord"

Consider a Riemannian metric

(r, θ) : coordinates

$$g = dr^2 + \{\varphi(r)\}^2 \, d\theta^2 \qquad \text{on} \qquad U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ and φ is a positive smooth function defined on $(0, r_0)$ with

$$\lim_{r \to +0} \varphi(r) = 0, \qquad \lim_{r \to +0} \varphi'(r) = 1.$$

Find a function φ such that (U, g) is flat.

$K = d\Omega + \Omega \wedge \Omega = 0$

(Hint: $[\partial/\partial r, (1/\varphi)\partial/\partial\theta]$ is an orthonormal frame.)

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2$$

$$\left(\frac{\partial}{\partial r} \perp \frac{\partial}{\partial \theta} \right) \quad \left|\frac{\partial}{\partial r}\right| = 1, \quad \left|\frac{\partial}{\partial \theta}\right| = \varphi$$

$$\omega^1 = dr \qquad \omega^2 = \varphi\, d\theta$$

$$e_1 := \frac{\partial}{\partial r} \qquad e_2 := \frac{1}{\varphi}\frac{\partial}{\partial \theta} \Rightarrow [e_1, e_2] : \text{orthonormal}$$

$$[e_1, e_1] = [e_2, e_2] = 0$$

$$[e_1, e_2] = \left[\frac{\partial}{\partial r}, \frac{1}{\varphi}\frac{\partial}{\partial \theta}\right] = -\frac{\varphi'}{\varphi^2}\frac{\partial}{\partial \theta} = -\frac{\varphi'}{\varphi} e_2$$

$$= -[e_2, e_1]$$

$$\omega_i^j(e_k) = \frac{1}{2}\Big(-\langle [e_i, e_j], e_k\rangle + \langle [e_j, e_k], e_i\rangle + \langle [e_k, e_i], e_j\rangle \Big),$$

$$\left(\omega_i^j = -\omega_j^i \right)$$

$$\omega_2^1(e_1) = \frac{1}{2}\Big(-\langle [e_2, e_1], e_1\rangle + \langle [e_1, e_1], e_2\rangle + \langle [e_1, e_2], e_1\rangle = 0$$

$$\omega_2^1(e_2) = \langle [e_1, e_2], e_2\rangle = -\frac{\varphi'}{\varphi}$$

$\omega^1_2(e_1) = 0 \qquad \omega^1_2(e_2) = -\frac{\varphi'}{\varphi}$

$$\boxed{\omega^1_2 = -\frac{\varphi'}{\varphi}\,\omega^2}$$

$$\Omega = \begin{bmatrix} 0 & \omega^1_2 \\ \omega^2_1 & 0 \end{bmatrix}$$

$\omega^2 \wedge \omega^2 = 0$

$$= -\frac{\varphi'}{\varphi}\,\omega_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$d\omega^2 = \sum_{s=1}^{2} \omega^s \wedge \omega^2_s$$

Lem 2.17

$$= \omega^1 \wedge \omega^2_1$$

$$= \frac{\varphi'}{\varphi}\,\omega^1 \wedge \omega^2$$

$$\boxed{K = d\Omega + \underbrace{\Omega \wedge \Omega}_{0}}$$

$$d\Omega = \left\{ -\left(\frac{\varphi'}{\varphi}\right)_r dr \wedge \omega^2 - \frac{\varphi'}{\varphi}\, d\omega^2 \right\} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \left\{ -\left(\frac{\varphi'}{\varphi}\right)_r - \left(\frac{\varphi'}{\varphi}\right)^2 \right\} \omega^1 \wedge \omega^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$K = d\Omega + \Omega \wedge \Omega = d\Omega$$

$$= \left\{ -\left(\frac{\varphi'}{\varphi}\right)' - \left(\frac{\varphi'}{\varphi}\right)^2 \right\} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \omega^1 \wedge \omega^2$$

$$= \left(-\frac{\varphi''}{\varphi} \right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \omega^1 \wedge \omega^2$$

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flat $(K = 0) \iff \varphi'' = 0$

$\iff \boxed{\varphi = r}$

$x = r\cos\theta$

$y = r\sin\theta$

$dx^2 + dy^2$

$$\boxed{g = dr^2 + r^2 d\theta^2}$$

the Euclidean metric on $\mathbb{R}^2$ w.r.to the polar coord

Problem (Ex. 3-2)

Compute the curvature form of $H^2(-c^2)$ with respect to an orthonormal frame $[e_1, e_2]$ as in Exercise 2-2

(identify)

$p_u = e_1, \quad p_v = \frac{1}{c} \sinh cu \; e_2$

"Parametrization"

$p_u = dp\left(\frac{\partial}{\partial u}\right) \sim \frac{\partial}{\partial u}$

$p_v = \frac{\partial}{\partial v}$

$\star \quad e_1 = \frac{\partial}{\partial u}, \quad e_2 = c \operatorname{cosech} cu \frac{\partial}{\partial v}$

$\Rightarrow \quad \omega^i_j, \; \kappa^i_j$: computable

Problem (Ex. 2-2)

Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and

$$H^2(-c^2) = \left\{ (x^0, x^2, x^2) \in \mathbb{R}^3_1 \,;\, \underline{\begin{array}{l} -(x^0)^2 + (x^1)^2 + (x^2)^2 = -\frac{1}{c^2}, \\ cx^0 > 0 \end{array}} \right\}$$

Verify that

$$\mathfrak{p} : (u, v) \mapsto \left(\frac{1}{c} \cosh cu, \frac{\cos v}{c} \sinh cu, \frac{\sin v}{c} \sinh cu \right)$$

gives a local coordinate system and

$$\begin{aligned} \boldsymbol{e}_1 &:= (\sinh cu, \cos v \cosh cu, \sin v \cosh cu), \\ \boldsymbol{e}_2 &:= (0, -\sin v, \cos v) \end{aligned}$$

orthonormal

forms a orthonormal frame.

Today's Goal : To define the sectional curvature
断面曲率

the curvature form $K = d\Omega + \Omega \wedge \Omega$

$$\kappa \quad K = (K_i^j) = \left(d\omega_i^j + \sum_l \omega_l^j \wedge \omega_i^l \right)$$

space of "constant curvature" ?

$\exists ?$ function(s) determined by (K_i^j)

2-form 1 form

$\alpha \in \Gamma(\wedge^2 T^*M)$, ω, $\mu \in \Gamma(T^*M)$

$\alpha(X, Y) = -\alpha(Y, X)$

$$\begin{aligned}(\omega \wedge \mu)(X, Y) &= \omega(X)\mu(Y) - \omega(Y)\mu(X),\\ (\alpha \wedge \omega)(X, Y, Z) &= (\omega \wedge \alpha)(X, Y, Z)\\ &:= \alpha(X, Y)\omega(Z) + \alpha(Y, Z)\omega(X) + \alpha(Z, X)\omega(Y).\end{aligned}$$

skew symmetric. "3 form"

$(\omega \wedge \mu) \wedge \lambda \sim \omega \wedge (\mu \wedge \lambda)$

1 form

Preliminaries: Differential Forms

$\alpha \in \Gamma(\wedge^2 T^*M)$, ω, $\mu \in \Gamma(T^*M)$, $f \in \mathcal{F}(M)$

$$df(X) = Xf$$

df: 1-form

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

dω: 2-form

$$\begin{aligned} d\alpha(X, Y, Z) = &X\alpha(Y, Z)d + Y\alpha(Z, X) + Z\alpha(X, Y) \\ &- \alpha([X, Y], Z) - \alpha([Z, X], Y) - \alpha([Y, Z], X). \end{aligned}$$

$$\underline{ddf = 0}, \qquad \underline{dd\omega = 0}, \qquad d(\mu \wedge \omega) = d\mu \wedge \omega - \mu \wedge d\omega.$$

- V: n-dimensional vector space with inner product $\langle\ ,\ \rangle$
- $[e_1, \ldots, e_n]$: an orthonormal basis

$$\wedge^2 V := \text{Span}\,\{\underbrace{e_i \wedge e_j}_{\text{orthonormal.}}, i < j\}$$

"formal exterior product"

$V \wedge V$

$\circledast$ $X \wedge Y \in \wedge^2 V$

$\Pi \subset V$: 2-dim subspace

$= \text{Span}\{X, Y\}$

$X \wedge Y \in \wedge^2 V$

(2-Grassmann of V)

$\wedge^2 V / \mathbb{R}_*$: the set of 2-dim subspaces