Advanced Topics in Geometry F (MTH.B502)

SEctional Curvature

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Addendum 1

Proposition (Prop 3.1; The local expression of the Lie bracket)

Let $(U; x^1, \dots, x^n)$ *be a coordinate neighborhood of an* n*-manifold* M*. Then the Lie bracket of two vector fields*

$$X = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j}, \qquad Y = \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}$$

is expressed as

$$[X, Y] = \sum_{j=1}^{n} \left(\xi^k \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \xi^j}{\partial x^k} \right) \frac{\partial}{\partial x^j}.$$

Addendum 2

Proposition (Prop 3.2)

Let U be a domain of a Riemannian n-manifold (M, g) and $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]$ an orthornomal frame on U. Then the connection form ω_i^j with respect to the frame $[\boldsymbol{e}_j]$ is obtained as

$$\omega_i^j(\boldsymbol{e}_k) = \frac{1}{2}\Big(- \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k \rangle + \langle [\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i \rangle + \langle [\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j \rangle \Big),$$

where $\langle\ ,\ \rangle$ denotes the inner product induced from g.

Exercise 3-1

Problem (Ex. 3-1)

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2 \, d\theta^2 \qquad \textit{on} \qquad U := \{(r,\theta)\, ;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r\to+0} \varphi(r) = 0, \qquad \lim_{r\to+0} \varphi'(r) = 1.$$

Find a function φ *such that* (U, g) *is flat.*
(Hint: $[\partial/\partial r, (1/\varphi)\partial/\partial\theta)]$ *is an orthonormal frame.)*

Exercise 3-2

Problem (Ex. 3-2)

Compute the curvature form of $H^2(-c^2)$ *with respect to an orthonormal frame* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *as in Exercise 2-2*

Exercise 2-2

Problem (Ex. 2-2)

Let $\mathbb{R}^3_1$ *be the* 3*-dimensional Lorentz-Minkowski space and*

$$H^2(-c^2) = \left\{ (x^0, x^2, x^2) \in \mathbb{R}^3_1 \,;\, \begin{matrix} -(x^0)^2 + (x^1)^2 + (x^2)^2 = -\frac{1}{c^2}, \\ cx^0 > 0 \end{matrix} \right\}$$

Verify that

$$(u, v) \mapsto \left(\frac{1}{c} \cosh cu, \frac{\cos v}{c} \sinh cu, \frac{\sin v}{c} \sinh cu \right)$$

gives a local coordinate system and

$$\begin{aligned} \boldsymbol{e}_1 &:= (\sinh cu, \cos v \cosh cu, \sin v \cosh cu), \\ \boldsymbol{e}_2 &:= (0, -\sin v, \cos v) \end{aligned}$$

forms a orthonormal frame.

Preliminaries: Differential Forms

$\alpha \in \Gamma(\wedge^2 T^*M)$, ω, $\mu \in \Gamma(T^*M)$

$$\begin{aligned}
(\omega \wedge \mu)(X, Y) &= \omega(X)\mu(Y) - \omega(Y)\mu(X), \\
(\alpha \wedge \omega)(X, Y, Z) &= (\omega \wedge \alpha)(X, Y, Z) \\
&:= \alpha(X, Y)\omega(Z) + \alpha(Y, Z)\omega(X) + \alpha(Z, X)\omega(Y).
\end{aligned}$$

Preliminaries: Differential Forms

$\alpha \in \Gamma(\wedge^2 T^*M)$, ω, $\mu \in \Gamma(T^*M)$, $f \in \mathcal{F}(M)$

$$\begin{aligned} df(X) &= Xf \\ d\omega(X,Y) &= X\omega(Y) - Y\omega(X) - \omega([X,Y]) \\ d\alpha(X,Y,Z) &= X\alpha(Y,Z)d + Y\alpha(Z,X) + Z\alpha(X,Y) \\ &\quad - \alpha([X,Y],Z) - \alpha([Z,X],Y) - \alpha([Y,Z],X). \end{aligned}$$

$$ddf = 0, \qquad dd\omega = 0, \qquad d(\mu \wedge \omega) = d\mu \wedge \omega - \mu \wedge d\omega.$$

Preliminaries: Exterior products

- V: n-dimensional vector space with inner product $\langle\ ,\ \rangle$
- $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]$: an orthonormal basis

$$\wedge^2 V := \mathsf{Span}\,\{\boldsymbol{e}_i \wedge \boldsymbol{e}_j, i < j\}$$