Advanced Topics in Geometry F (MTH.B502)

SEctional Curvature

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-f/

Tokyo Institute of Technology

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Sectional Curvature

Definition (Definition 4.5)

Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in T_pM. The sectional curvature of (M, g) with respect to the plane Π_p is a number

sectional curvature

$$K(\Pi_p) := \underline{\boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w})},$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of Π_p

$$K : \bigcup_p (\underline{\text{2-dim subspaces of } T_pM}) \longrightarrow \mathbb{R}$$

$$\bigcup_p Gr_2(T_pM) \quad \text{2-Grassmanian}$$

$$\shortparallel$$

$$Gr_2(TM)$$

Gaussian curvature

$\dim M = 2$

1 point

$K : M \to \mathbb{R}$

Curvature forms

- (M, g): a Riemannian n-manifold.
- $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$: an orthonormal frame on $U \subset M$.
- (ω^j): the dual frame $\omega^j(e_k) = \delta^j_k$
- $\Omega = (\omega_i^j)$: the connection form.
- $K = (\kappa_i^j) = d\Omega + \Omega \wedge \Omega$: the curvature form.

$$d\omega^i = \sum_s \omega^s \wedge \omega_s^i,$$

$$\kappa_j^i = d\omega_j^i + \sum_s \omega_s^i \wedge \omega_j^s$$

A bilinear form induced from the curvature form

$p \in U$: fix $\quad \theta_k = \theta_k(p)$

$$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j,k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l)\xi^{kl}\eta^{ij} \quad \in \mathbb{R}$$

$$\boldsymbol{\xi} = \sum_{k<l} \xi^{kl} \boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j} \eta^{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j \quad \in \Lambda^2(T_pM)$$

- $\boldsymbol{K}$ is a bilinear form on $\boxed{\wedge^2 T_pM}$

$\sum_{j=1}^{3} \Theta_j^i \Theta_j^k = \delta^{ik}$

independent of choice of $[\theta_j]$

$$[\theta_1 \cdots \theta_n] = [\vartheta_1 \cdots \vartheta_n]\Theta$$

$$\Theta = (\Theta_j^i) \in O(n) \quad ({}^t\Theta = \Theta^{-1})$$

A bilinear form induced from the curvature form

$p \in U$: fix

$$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j, k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l)\xi^{kl}\eta^{ij},$$

$$\boldsymbol{\xi} = \sum_{k<l} \xi^{kl} \boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j} \eta^{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

Lemma (Lemma 4.4)

$\boldsymbol{K}$ *is symmetric.*

$\mathbb{K}(\xi, \eta) = \mathbb{K}(\eta, \xi)$

$\Rightarrow$ $\mathbb{K}(\xi, \xi)$ determines $\mathbb{K}$

diagonal components

$2\mathbb{K}(\xi, \eta)$

$= \mathbb{K}(\xi+\eta, \xi+\eta)$

$- \mathbb{K}(\xi, \xi) - \mathbb{K}(\eta, \eta)$

Proof of Lemma 4.4 (1)

$$\kappa^i_s = d\omega^i_s + \sum_t \omega^i_t \wedge \omega^t_s$$

Proposition (The first Bianchi identity; Prop. 4.2)

$$\kappa^i_j(e_k, e_l) + \kappa^i_k(e_l, e_j) + \kappa^i_l(e_j, e_k) = 0.$$

$$d\omega^i = \sum_{s=1}^n \omega^s \wedge \omega^i_s$$

$$0 = dd\omega^i = \sum \left(d\omega^s \wedge \omega^i_s - \omega^s \wedge d\omega^i_s \right)$$

$$= \sum_{s,t} \Big(\cancel{\omega^t \wedge \omega^s_t \wedge \omega^i_s} - \omega^s \wedge \kappa^i_s + \cancel{\omega^s \wedge \omega^i_t \wedge \omega^t_s} \Big)$$

$$\underbrace{\sum_s \omega^s \wedge \kappa^i_s}_{3\text{ form}} = 0$$

$$(e_j,\ e_k,\ e_l$$

Rem

$$\kappa^i_j(e_k, e_l) = \langle R(e_k, e_l) e_j, e^i \rangle$$

the Riemann curvature tensor

Proof of Lemma 4.4 (2)

Corollary (Cor. 4.3)

$\kappa^i_j(e_k, e_l) = \kappa^k_l(e_i, e_j).$

$\kappa^i_j = -\kappa^j_i$

$\kappa^i_j(e_k, e_l) + \kappa^i_k(e_l, e_j) + \kappa^i_l(e_j, e_k) = 0.$

$$\kappa^j_k(e_i, e_l) + \kappa^j_i(e_l, e_k) + \kappa^j_l(e_k, e_i) = 0$$

$$\kappa^k_i(e_j, e_l) + \kappa^k_j(e_l, e_i) + \kappa^k_l(e_i, e_j) = 0$$

$+$

$$= 0$$

$$\kappa^i_j(e_k, e_l) + \kappa^i_k(e_l, e_j) + \kappa^i_k(e_l, e_i) = 0$$

... $\boxed{\text{etc}}$

Proof of Lemma 4.4 (3)

$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j,k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l)\xi^{kl}\eta^{ij}$

Corollary (Cor. 4.3)

$\kappa_j^i(\boldsymbol{e}_k, \boldsymbol{e}_l) = \kappa_l^k(\boldsymbol{e}_i, \boldsymbol{e}_j).$

Lemma (Lemma 4.4)

$\boldsymbol{K}$ *is symmetric.*

Sectional Curvature

Definition (Definition 4.5)

Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in T_pM. The sectional curvature of (M, g) with respect to the plane Π_p is a number

$$\boxed{K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),}$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of Π_p

Exercise 4-1

Problem (Ex. 4-1)

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \text{ on } U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r\to+0} \varphi(r) = 0, \qquad \lim_{r\to+0} \frac{\varphi(r)}{r} = 1.$$

Classify the function φ *so that* g *is of constant sectional curvature.*

$$\left(\begin{array}{l} K_1^2 = K\, \omega^1\wedge\omega^2 \\ K_2^1 ? \qquad \uparrow \text{sectional curvature} \end{array} \right)$$

Exercise 4-2

Problem (Ex. 4-2)

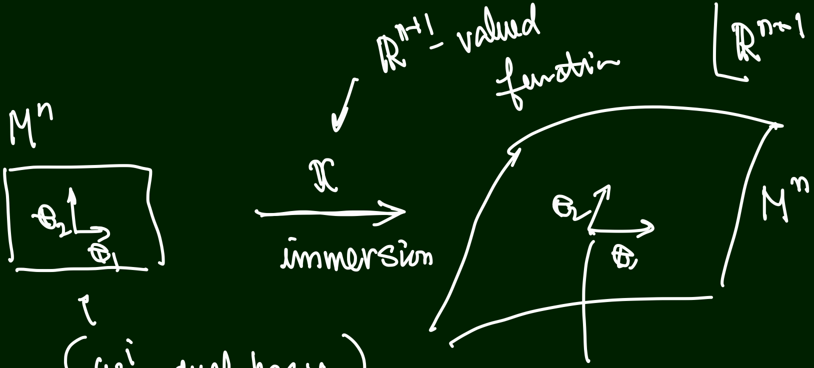
Let $M \subset \mathbb{R}^{n+1}$ *be an embedded submanifold with the Riemannian metric induced from the canonical Euclidean metric of* $\mathbb{R}^{n+1}$. *Then the position vector vector* $\boldsymbol{x}(p)$ *of* $p \in M$ *induces a smooth map*

$$\boldsymbol{x} \colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$*-tuple of* C^∞*-functions. Let* $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]$ *be an orthonormal frame defined on a domain* $U \subset M$. *Since* $T_pM \subset \mathbb{R}^{n+1}$, *we can consider that* $\boldsymbol{e}_j$ *is a smooth map from* $U \to \mathbb{R}^{n+1}$. *Take a dual basis* (ω^j) *to* $[\boldsymbol{e}_j]$. *Prove that*

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j$$

holds on U.



$(\omega^i$ dual basis$)$

$\mathbb{R}^{n+1}$ - valued function on M

$$dx = \sum_{i=1}^{n} \omega^i e_i$$

$\mathbb{R}^{n+1}$- valued 1-form