Advanced Topics in Geometry F (MTH.B502)

SEctional Curvature

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Sectional Curvature

Definition (Definition 4.5)

Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in T_pM. The sectional curvature of (M, g) with respect to the plane Π_p is a number

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of Π_p

Curvature forms

- (M, g): a Riemannian n-manifold.
- $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$: an orthonormal frame on $U \subset M$.
- (ω^j): the dual frame
- $\Omega = (\omega_i^j)$: the connection form.
- $K = (\kappa_i^j) = d\Omega + \Omega \wedge \Omega$: the curvature form.

$$d\omega^i = \sum_s \omega^s \wedge \omega_s^i,$$

$$\kappa_j^i = d\omega_j^i + \sum_s \omega_s^i \wedge \omega_j^s$$

A bilinear form induced from the curvature form

$p \in U$: fix

$$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j, k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l) \xi^{kl} \eta^{ij},$$

$$\boldsymbol{\xi} = \sum_{k<l} \xi^{kl} \boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j} \eta^{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

- $\boldsymbol{K}$ is a bilinear form on $\wedge^2 T_p M$

A bilinear form induced from the curvature form

$p \in U$: fix

$$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j, k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l) \xi^{kl} \eta^{ij},$$

$$\boldsymbol{\xi} = \sum_{k<l} \xi^{kl} \boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j} \eta^{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

Lemma (Lemma 4.4)

$\boldsymbol{K}$ *is symmetric.*

Proof of Lemma 4.4 (1)

Proposition (The first Bianchi identity; Prop. 4.2)

$\kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) + \kappa^i_k(\boldsymbol{e}_l, \boldsymbol{e}_j) + \kappa^i_l(\boldsymbol{e}_j, \boldsymbol{e}_k) = 0.$

Proof of Lemma 4.4 (2)

Corollary (Cor. 4.3)

$\kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) = \kappa^k_l(\boldsymbol{e}_i, \boldsymbol{e}_j)$.

Proof of Lemma 4.4 (3)

$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j,k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l)\xi^{kl}\eta^{ij}$

Corollary (Cor. 4.3)

$\kappa_j^i(\boldsymbol{e}_k, \boldsymbol{e}_l) = \kappa_l^k(\boldsymbol{e}_i, \boldsymbol{e}_j)$.

Lemma (Lemma 4.4)

$\boldsymbol{K}$ *is symmetric.*

Sectional Curvature

Definition (Definition 4.5)

Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in T_pM. The <u>sectional curvature</u> of (M, g) with respect to the plane Π_p is a number

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of Π_p

Exercise 4-1

Problem (Ex. 4-1)

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2 \, d\theta^2 \text{ on } U := \{(r, \theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r \to +0} \varphi(r) = 0, \qquad \lim_{r \to +0} \frac{\varphi(r)}{r} = 1.$$

Classify the function φ *so that* g *is of constant sectional curvature.*

Exercise 4-2

Problem (Ex. 4-2)

Let $M \subset \mathbb{R}^{n+1}$ be an embedded submanifold with the Riemannian metric induced from the canonical Euclidean metric of $\mathbb{R}^{n+1}$. Then the position vector vector $\boldsymbol{x}(p)$ of $p \in M$ induces a smooth map

$$\boldsymbol{x}\colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$-tuple of C^∞-functions. Let $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ be an orthonormal frame defined on a domain $U \subset M$. Since $T_pM \subset \mathbb{R}^{n+1}$, we can consider that $\boldsymbol{e}_j$ is a smooth map from $U \to \mathbb{R}^{n+1}$. Take a dual basis (ω^j) to $[\boldsymbol{e}_j]$. Prove that

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j$$

holds on U.