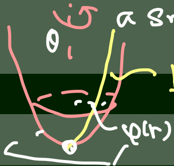
Advanced Topics in Geometry F (MTH.B502)

Space forms

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-f/

Tokyo Institute of Technology

2022/07/19



a surface of revolution

r (arclength parameter of the generating curve)

Problem (Ex. 4-1)

Consider a Riemannian metric

warped product metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \text{ on } U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r\to+0} \varphi(r) = 0, \qquad \lim_{r\to+0} \frac{\varphi(r)}{r} = 1.$$

Classify the function φ *so that* g *is of constant sectional curvature.*

$K = -\dfrac{\varphi''}{\varphi}$

$$\boxed{\kappa_1^2 = -\frac{\varphi''}{\varphi}\omega^1 \wedge \omega^2}$$

$e_1 = \dfrac{\partial}{\partial r}$ $\quad e_2 = \dfrac{1}{\varphi}\dfrac{\partial}{\partial \theta}$

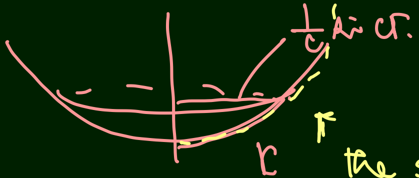
$\omega^1 = dr$ $\quad \omega_2 = \varphi\, d\theta$

$$K = -\frac{\varphi''}{\varphi}$$

$K = \text{const.}$

$K = 0 \Rightarrow \varphi(r) = r \quad (\text{Ex } 3\text{-}1)$

$K = c^2 > 0 \qquad \varphi'' = -c^2\varphi$



$$\varphi = A\cos cr + B\sin cr$$

$$= \frac{1}{c}\sin cr.$$

the sphere of radius $\frac{1}{c}$]

$K = -c^2 < 0 \qquad \varphi = A\cosh cr + B\sinh cr$

$\nexists$ corresponding surface of revolution in $\mathbb{R}^3$ $\quad = \frac{1}{c}\sinh cr$

Exercise 4-2

Problem (Ex. 4-2)

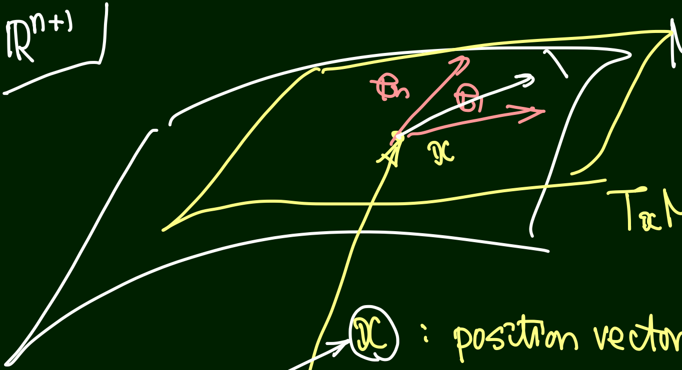
Let $M \subset \mathbb{R}^{n+1}$ *be an embedded submanifold with the Riemannian metric induced from the canonical Euclidean metric of* $\mathbb{R}^{n+1}$*. Then the position vector vector* $\boldsymbol{x}(p)$ *of* $p \in M$ *induces a smooth map*

$$\boldsymbol{x}\colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$*-tuple of* C^∞*-functions. Let* $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ *be an orthonormal frame defined on a domain* $U \subset M$*. Since* $T_pM \subset \mathbb{R}^{n+1}$*, we can consider that* $\boldsymbol{e}_j$ *is a smooth map from* $U \to \mathbb{R}^{n+1}$*. Take a dual basis* (ω^j) *to* $[\boldsymbol{e}_j]$*. Prove that*

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j$$

holds on U*.*



M^n : hypersurface (超曲面)

$T_xM \subset T_x\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$

x : position vector

$[e_j]$: orthonormal frame

$\mathbb{R}^{n+1}$-valued function on M.

$(\omega^j)_{j=1}^{n}$: dual (1 forms on M)

$\omega^j(e_k) = \delta^j_k$

$$dx = \sum_j \omega^j e_j$$

derivative as $(n+1)$-tuple of functions

dx : $(n+1)$-tuple of 1-form i.e. $\mathbb{R}^{n+1}$-valued 1-form

$dx(X) = Xx$ = the directional derivative of x w.r.t. X

$(X \in T_pM)$

$= X \in T_pM \subset \mathbb{R}^{n+1}$

$$\therefore \quad dx(e_{\cancel{j}e}) = e_{\cancel{j}e}$$

$$dx = \sum_{j\geq 1}^{n} \omega^j e_j$$

$\boxed{\text{space of constant sectional curvature}}$