Advanced Topics in Geometry F (MTH.B502)

Space forms

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-f/

Tokyo Institute of Technology

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Exercise 4-1

Problem (Ex. 4-1)

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \text{ on } U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r\to+0} \varphi(r) = 0, \qquad \lim_{r\to+0} \frac{\varphi(r)}{r} = 1.$$

Classify the function φ *so that* g *is of constant sectional curvature.*

$$\kappa_1^2 = -\frac{\varphi''}{\varphi}\omega^1 \wedge \omega^2$$

Exercise 4-2

Problem (Ex. 4-2)

Let $M \subset \mathbb{R}^{n+1}$ be an embedded submanifold with the Riemannian metric induced from the canonical Euclidean metric of $\mathbb{R}^{n+1}$. Then the position vector vector $\boldsymbol{x}(p)$ of $p \in M$ induces a smooth map

$$\boldsymbol{x} \colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$-tuple of C^∞-functions. Let $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ be an orthonormal frame defined on a domain $U \subset M$. Since $T_pM \subset \mathbb{R}^{n+1}$, we can consider that $\boldsymbol{e}_j$ is a smooth map from $U \to \mathbb{R}^{n+1}$. Take a dual basis (ω^j) to $[\boldsymbol{e}_j]$. Prove that

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j$$

holds on U.