Advanced Topics in Geometry F (MTH.B502)

Space forms

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Constant sectional curvature 1 (pointwise)

Theorem (Thm. 5.1)

Assume there exists a real number k such that $K(\Pi_p) = k$ for all 2-dimensional subspace $\Pi_p \in T_pM$ for a fixed p. Then the curvature form is expressed as

the sectional curv. does not depend of choice of planes

$$\boxed{\kappa_i^j = k\omega^i \wedge \omega^j.}$$

$\hat{i}$ $\hat{j}$

Conversely, the curvature form is written as above, the sectional curvature at p is constant k.

constant

$p \in M$: fix

$K : \underbrace{Gr_2(T_pM)}_{\text{the set of 2-dim subspace in } T_pM} \longrightarrow \mathbb{R}$

Constant sectional curvature 1

- $k = K(\mathrm{Span}\{\boldsymbol{e}_i, \boldsymbol{e}_j\}) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_j, \boldsymbol{e}_i \wedge \boldsymbol{e}_j) = \kappa^i_j(\boldsymbol{e}_i, \boldsymbol{e}_j)$.
- $\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = 0$.
- $\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = 0$.

$$\boxed{\kappa^j_i(\boldsymbol{e}_k, \boldsymbol{e}_l) = \begin{cases} k & (i,j) = (k,l) \\ 0 & \text{otherwise}, \end{cases}}$$

$\underline{\kappa^i_j(e_k\ e_l)}$ $= 0$ if $(i, j) \neq (k, l)$

$\kappa^i_j = \sum_{k<l} a_{kl}\, \omega^k \wedge \omega^l$

1st Bianchi

$$\boxed{\kappa^i_j = k\, \omega^i \wedge \omega^j}$$

Constant sectional curvature 2 (global)

Theorem (Thm. 5.2)

Assume that for each p, there exists a real number $k(p)$ such that $\boxed{K(\Pi_p) = k(p)}$ for any $\Pi_p \in \mathrm{Gr}_2(T_pM)$. Then the function $k\colon M \ni p \mapsto k(p) \in \mathbb{R}$ is constant provided that M is connected.

$$\boxed{d\kappa_i^j = \sum_s \left(\kappa_s^j \wedge \omega_i^s - \omega_s^j \wedge \kappa_i^s\right),}$$

← the 2nd Bianchi identity

$$d\kappa_i^j = \cancel{dd\omega_i^j} + d\left(\sum_{s=1}^n \omega_s^j \wedge \omega_i^s\right)$$

$$d\omega^j = \sum_s \omega^s \wedge \omega_s^j$$

$$d\omega_s^j = \kappa_s^j - \sum_t \omega_t^j \wedge \omega_s^t$$

$$\boxed{\kappa_j^i = k(p)\,\omega^i \wedge \omega^j} \Longrightarrow \boxed{dk = 0} \Rightarrow k = \text{const}$$

Space Forms

Definition (Def. 5.3) 空間型

An n-dimensional space form is a complete Riemannian n-manifold M of constant sectional curvature.

complete: $\forall$ divergent path has ∞-length.

M

- $\gamma : [0, +\infty) \to M$
- $\forall C \subset M$ compact, $\exists\, t_0 \in [0, +\infty)$ s.t. $\gamma([t_0, \infty)) \subset M \setminus C$

M = an open disc

ds^2: ordinary

incomplete

a compact Riem. manifold is complete

completeness: usually defined in terms of "geodesics"

our definition: a conclusion of the Hopf-Rinow theorem

The Euclidean n-space: $\mathbb{R}^n$ $K = (\kappa_i^j) = O$.

constant sectional curvature 0

- completeness : $\gamma : [0, \infty) \to \mathbb{R}^n$: a divergent path

$\Rightarrow \lim_{t \to +\infty} |\gamma(t)| = +\infty$

$\forall r > 0 \ \exists t_0 > 0$

$\gamma([t_0, +\infty)) \subset \mathbb{R}^n \setminus \overline{B(0,r)}$

$\boxed{|\gamma| \underset{[t_0, +\infty)}{>} r}$

$\Rightarrow \lim_{t \to +\infty} \int_0^t |\dot\gamma(\tau)| d\tau$

$\geq \lim \left| \int_0^t \dot\gamma(\tau) d\tau \right|$

$= \lim |\gamma(t) - \gamma(0)| \geq \lim |\gamma(t)| - |\gamma(0)| = +\infty$

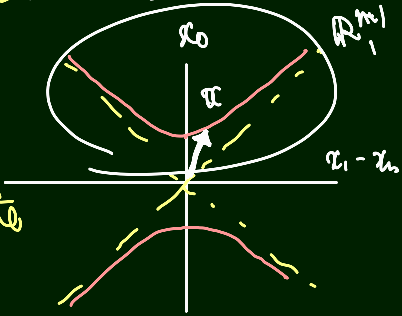
Lorentz Minkowski space

$$H^n(-c^2) := \left\{ \boldsymbol{x} = (x^0, \ldots, x^n) \in \mathbb{R}^{n+1}_1 \;\middle|\; \langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -\frac{1}{c^2}, cx_0 > 0 \right\},$$

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2$$

$$T_x H^n(-c^2) = x^{\perp} = \{ v \mid \langle x, v \rangle_L = 0 \}$$

$g_H = \langle \, , \, \rangle_L \big|_{T_x H^n(-c^2)}$: positive definite



$(H^n(-c^2), g_H)$: Riem. mfd

the hyperboloid of 2-sheets = 二葉双曲面

Theorem

The hyperbolic space $(H^n(-c^2), g_H)$ is of constant sectional curvature $-c^2$

$(e_1 \cdots e_n)$: an orthonormal frame dual (ω^i)

the x: position vector vector-valued

$dx = \sum \omega^i e_i$

$\mathcal{F} := (e_0 \cdots e_n)$

$e_0 = cx$ $\langle e_0, e_0 \rangle = -1$

a pseudo orthonormal frame of $\mathbb{R}^{n+1}_1$

$$d\theta_0 = c\sum w^i \theta_i = \sum c w^i \theta_i$$

$$d\theta_j = -\overset{cw^i}{h_j}\,\theta_0 + \sum \alpha_j^k \theta_k$$

$$\langle d\theta_j, \theta_0\rangle_L = -h_j$$

$$\|$$

$$\underbrace{\cancel{d\langle \theta_j, \theta_0\rangle_L}}_{0} - \langle \theta_j, d\theta_0\rangle_L = -c w^{\cancel{i}\,j}$$

$$h_j = c w^j$$

$$0 = dde_0 = d\left(\sum \omega^i e_i\right)$$

$$= \sum_i \left(d\omega^i\, e_i - \omega^i \wedge de_i\right)$$

$$= \left(\sum_{i,s} \left(\omega^s \wedge \omega^i_s\right) e_i - \sum_{s,i} \left(\omega^s \wedge \alpha^i_s\right) e_i\right)$$

$$\cancel{-\left(\omega^i \wedge \omega^i\right) e_0}$$

$$= \sum \left(\omega^s \wedge \left(\omega^i_s - \alpha^i_s\right)\right) e_i = 0$$

skew-symmetric

$$\Rightarrow \boxed{\alpha^i_s = \omega^i_s}$$

$$d e_0 = \sum_{j=1}^{n} c\,\omega^j e_j$$

$$d e_j = c\,\omega^j e_0 + \sum \omega^k_j e_k$$

$$\mathcal{F} = (e_0 \; \cdots \; e_n)$$

$$d\mathcal{F} = \mathcal{F}\tilde{\Omega} = \mathcal{F}\begin{bmatrix} 0 & c\omega^1 & \cdots & c\omega^n \\ c\omega^1 & & & \\ \vdots & & \Omega & \\ c\omega^n & & & \end{bmatrix}$$

integrability condition

$$\boxed{K^i_j = -c^2\,\omega^i \wedge \omega^j}$$

Exercise 5-1

Problem (Ex. 5-1)

Prove that the sphere

$$S^n(c^2) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \, ; \, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{c^2} \right\}$$

of radius $1/c$ *in the Eucidean* $n+1$*-space is of constant sectional curvature* c^2.

Exercise 5-2

Problem (Ex. 5-2)

Let $f\colon U \to \mathbb{R}^{n+1}$ be an immersion defined on a domain $U \subset \mathbb{R}^n$, and ν a unit normal vector field. Take an orthornormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ of the tangent bundle of U, and consider each $\boldsymbol{e}_j$ a map into $\mathbb{R}^{n+1}$. In addition, we consider ν an $\mathbb{R}^{n+1}$-valued function. Prove that

$$d\nu = -\sum_j h^j \boldsymbol{e}_j, \qquad \textit{where} \qquad h^j := \langle d\boldsymbol{e}_j, \nu\rangle .$$