Advanced Topics in Geometry F (MTH.B502)

Space forms

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Constant sectional curvature 1

Theorem (Thm. 5.1)

Assume there exists a real number k such that $K(\Pi_p) = k$ for all 2-dimensional subspace $\Pi_p \in T_pM$ for a fixed p. Then the curvature form is expressed as

$$\kappa_i^j = k\omega^i \wedge \omega^j.$$

Conversely, the curvature form is written as above, the sectional curvature at p is constant k.

Constant sectional curvature 1

- $k = K(\mathrm{Span}\{\boldsymbol{e}_i, \boldsymbol{e}_j\}) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_j, \boldsymbol{e}_i \wedge \boldsymbol{e}_j)] = \kappa^i_j(\boldsymbol{e}_i, \boldsymbol{e}_j)$.
- $\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = 0$.
- $\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = 0$.

$$\kappa^j_i(\boldsymbol{e}_k, \boldsymbol{e}_l) = \begin{cases} k & (i,j) = (k,l) \\ 0 & \text{otherwise,} \end{cases}$$

Constant sectional curvature 2

Theorem (Thm. 5.2)

Assume that for each p, there exists a real number $k(p)$ such that $K(\Pi_p) = k(p)$ for any $\Pi_p \in \mathrm{Gr}_2(T_pM)$. Then the function $k\colon M \ni p \to k(p) \in \mathbb{R}$ is constant provided that M is connected.

$$d\kappa_i^j = \sum_s \left(\kappa_s^j \wedge \omega_i^s - \omega_s^j \wedge \kappa_i^s\right),$$

Space Forms

Definition (Def. 5.3)

An n-dimensional space form is a complete Riemannian n-manifold of constant sectional curvature.

The Euclidean space

The Euclidean n-space: $K = (\kappa_i^j) = O$.

The Hyperbolic space

$$H^n(-c^2) := \left\{ \boldsymbol{x} = (x^0, \dots, x^n) \in \mathbb{R}^{n+1}_1 \;\middle|\; \langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -\frac{1}{c^2}, cx_0 > 0 \right\},$$

The Hyperbolic space

Theorem

The hyperbolic space $(H^n(-c^2), g_H)$ *is of constant sectional curvature* $-c^2$.

Exercise 5-1

Problem (Ex. 5-1)

Prove that the sphere

$$S^n(c^2) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{c^2} \right\}$$

of radius $1/c$ *in the Eucidean* $n+1$*-space is of constant sectional curvature* c^2*.*

Exercise 5-2

Problem (Ex. 5-2)

Let $f : U \to \mathbb{R}^{n+1}$ *be an immersion defined on a domain* $U \subset \mathbb{R}^n$*, and* ν *a unit normal vector field. Take an orthornormal frame* $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ *of the tangent bundle of* U*, and consider each* $\boldsymbol{e}_j$ *a map into* $\mathbb{R}^{n+1}$*. In addition, we consider* ν *an* $\mathbb{R}^{n+1}$*-valued function. Prove that*

$$d\nu = -\sum_j h^j \boldsymbol{e}_j, \qquad \text{where} \qquad h^j := \langle d\boldsymbol{e}_j, \nu \rangle .$$