Advanced Topics in Geometry F (MTH.B502)

Local uniqueness of space forms

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Problem (Ex. 5-1)

Prove that the sphere

Euclidean sp.

$$S^n(c^2) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{c^2} \right\}$$

the sphere centered at the origin with radius $\frac{1}{c}$

of radius $1/c$ in the Eucidean $(n+1)$-space is of constant sectional curvature c^2.

Corrections on Proof of Theorem 5.5: $\frac{1}{c} \to c$

$$H^n(-c^2) = \left\{ x \in \mathbb{R}^{n+1}_1 \,;\, \langle x, x \rangle_L = -\frac{1}{c^2},\ cx^0 > 0 \right\}$$

$$K = -c^2$$

• Take an orthonormal frame $[e_1 \cdots e_n]$ on a nbd U of $p \in S^n$

$x : S^n \to \mathbb{R}^{n+1}$ the position vector.

$U \subset \mathbb{R}^{n+1}$

• $e_0 = c\,x$

$$T_x S^n = x^{\perp} \subset \mathbb{R}^{n+1}$$

$(n+1)\times(n+1)$ $SO(n+1)$

• $\mathcal{F} = (e_0, e_1, \cdots, e_n) : U \xrightarrow[C^\infty]{} \cancel{M_{n+1}(\mathbb{R})}$

• $d\mathcal{F} = \mathcal{F}\widetilde{\Omega}$; $\widetilde{\Omega} = \left(\begin{array}{c|c} 0 & -c\,{}^t\omega \\ \hline c\,\omega & \Omega \end{array}\right)$

$\omega = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}$ 1 form on U. : the dual frame of $[e_j]$

$\Omega = (\omega^i_j)$ the connection form

In fact

✓ $d\mathbf{e}_0 = c\,dx = c(\omega^1 \mathbf{e}_1 + \cdots + \omega^n \mathbf{e}_n)$

$\langle d\mathbf{e}_j, \mathbf{e}_0 \rangle = \cancel{d\langle \mathbf{e}_j, \mathbf{e}_0\rangle} - \langle \mathbf{e}_j, d\mathbf{e}_0 \rangle \quad j \geq 1$

$= -c\omega^j$

So one can set

$d\mathbf{e}_j = -c\omega^j \mathbf{e}_0 + \sum_{\ell=1}^{n} \alpha_j^\ell \mathbf{e}_\ell$ $\quad \omega_j^\ell$

$+$ for the case of $H^n(-c^2)$ because $\langle \mathbf{e}_0, \mathbf{e}_0 \rangle = -1$

$\alpha_j^\ell = \langle d\mathbf{e}_j, \mathbf{e}_\ell \rangle$

$d(f\omega) = df \wedge \omega + f d\omega$

$0 \overset{\checkmark}{=} \frac{1}{c} dd\mathbf{e}_0 = d\left(\sum_{s=1}^{n} \omega^s \mathbf{e}_s\right) = \sum_s d\omega^s \mathbf{e}_s \; \cancel{+} \; - \sum \omega^s \wedge d\mathbf{e}_s$

$$0 = \sum_s d\omega^s e_s - \sum_s \omega^s \wedge de_s$$

$$= \sum_{u,s} \omega^u \wedge \omega^s_u \, e_s - \sum_{s,u} \omega^{\cancel{s}\,u} \wedge \alpha^{\cancel{u}\,s}_{\cancel{s}\,u} \, e_{\cancel{u}\,s}$$

$$+ \cancel{\sum_s \omega^s \wedge c\,\omega^s}$$

$$\sum_u \omega^u \wedge \omega^s_u = \sum_u \omega^u \wedge \alpha^s_u$$

$$\underline{\omega^s_u = -\omega^u_s} \quad ; \quad \underline{\alpha^s_u} = \langle de_u, e_s \rangle$$

$$= -\langle e_u, de_s \rangle = \underline{-\alpha^u_s}$$

$$\Rightarrow \quad \omega^s_u = \alpha^s_u$$

Summing up: $dF = F\tilde{\Omega}$ $\quad \tilde{\Omega} = \begin{pmatrix} 0 & -c\,\omega^t \\ c\,\omega & \Omega \end{pmatrix}$

$$d\mathcal{F} = \mathcal{F}\tilde{\Omega}$$

$$\Rightarrow \quad d\tilde{\Omega} + \tilde{\Omega} \wedge \tilde{\Omega} = 0 \quad \text{(integrability)}$$

$$\Rightarrow \quad \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \overset{(3)}{=} \begin{pmatrix} 0 & 0 \\ 0 & c^2 \omega \wedge {}^t\omega \end{pmatrix}$$

$$K^i_j = c^2 \omega^i \wedge \omega^j$$

sect. curv : c^2

Exercise 5-2

Problem (Ex. 5-2)

immersed submanifold (hypersurface)

Let $f : U \to \mathbb{R}^{n+1}$ be an immersion defined on a domain $U \subset \mathbb{R}^n$, and ν a unit normal vector field. Take an orthornormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ of the tangent bundle of U, and consider each $\boldsymbol{e}_j$ a map into $\mathbb{R}^{n+1}$. In addition, we consider ν an $\mathbb{R}^{n+1}$-valued function. Prove that

$$d\nu = -\sum_j h^j \boldsymbol{e}_j, \quad \text{where} \quad h^j := \langle d\boldsymbol{e}_j, \nu \rangle .$$

2nd fundamental form

$U^n \subset \mathbb{R}^{n+1}$ hypersurface

$[e_1 \cdots e_n]$ orthonormal frame

$\nu = e_{n+1}$; $\nu \perp e_j$ $(j = 1, \cdots n)$, $|\nu| = 1$

$$d\nu = \sum_{j=0}^{\cancel{n+1}\; n} \eta^j e_j \qquad \eta^j : \text{1-form}$$

vector valued function

$$\overset{\nu_n}{[e_1 \cdots e_n,\ e_{n+1}]}$$

orthonormal basis

$$\eta^{n+1} = \langle d\nu, e_{n+1} \rangle$$

$$= \langle d\nu, \nu \rangle = \frac{1}{2} d\langle \nu, \nu \rangle = 0$$

$j = 1 \cdots n$

$$\eta^j = \langle d\nu, e_j \rangle$$

$$= \cancel{d\langle \nu, e_j \rangle} - \langle \nu, de_j \rangle$$