Advanced Topics in Geometry F (MTH.B502)

Local uniqueness of space forms

Riem mfd of constant sectional curvature.

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Definition (Def. 6.2)

h

A C^∞-map $f\colon M \to N$ between Riemannian manifolds (M, g) and (N, h) is called a local isometry if $\dim M = \dim N$ and $f^*h = g$ hold, that is,

tangent vectors on M

pull back of h by f

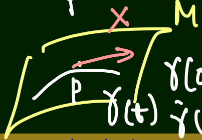
$$f^*h(X, Y) := h(df(X), df(Y)) = g(X, Y)$$

isometry

holds for X, $Y \in T_pM$ and $p \in M$.

- A local isometry is an immersion (Lemma 6.3)

$$(df)_p : T_pM \longrightarrow T_{f(p)}N \quad \text{linear}$$



$\gamma(0) = p$

$\dot\gamma(0) = X$

$$(df)_p(X) = \frac{d}{dt}\Big|_0 (f \circ \gamma)$$

Isometry

Corollary

A smooth map $f\colon (M,g) \to (N,h)$ is a local isometry if and only if for each $p \in M$,

$$[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] := [df(\boldsymbol{e}_1), \ldots, df(\boldsymbol{e}_n)]$$

is an orthonormal frame for some orthonormal frame $[\boldsymbol{e}_j]$ on a neighborhood of p.

$[\boldsymbol{e}_1 \; \cdots \; \boldsymbol{e}_n]$: an orthonormal frame on $U \subset M$

Local Uniqueness Theorem

Theorem (Thm. 6.5)

Let $U \subset \mathbb{R}^n$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (U, g) is constant k, there exists a local isometry $f \colon U \to N^n(k)$, where

$$N^n(k) = \begin{cases} S^n(k) & (k > 0) \\ \mathbb{R}^n & (k = 0) \\ H^n(k) & (k < 0). \end{cases}$$

(U, g) : const sect. curvature

$\Rightarrow$ (locally) U can be considered as a domain in $N^n(k)$.

Theorem 6.5; $k = 0$

$[\mathbf{e}_1 \cdots \mathbf{e}_n]$: an orthonormal frame on (U, g)

$(\omega^1 \cdots \omega^n)$: the dual of $[\mathbf{e}_j]$

$\Omega = (\omega^i_j)$: the connection form

$\Rightarrow K = d\Omega + \Omega \wedge \Omega = O$ ♡

skew symmetric ${}^t\Omega + \Omega = O$

Consider

$(*)$ $dF = F\Omega$; $F(p_0) = \mathrm{id}$

By ♡, $\exists F : U \xrightarrow{C^\infty} \cancel{M_n(\mathbb{R})}$ $SO(n)$ with $(*)$

$F = [\mathbf{v}_1, \cdots \mathbf{v}_n]$: orthonormal.

$$\alpha := \omega^1 v_1 + \cdots + \omega^n v$$

__Claim__ $d\alpha = 0 \Rightarrow \exists\, f : U \to \mathbb{R}^n$ s.t. $df = \alpha$

Poincaré lemma

$$df = \omega^1 v_1 + \cdots + \omega^n v_n$$

$$\Rightarrow df(\theta_j) = v_j \quad \square \qquad\qquad \omega^j(\theta_l) = \delta^j_l$$

$$d\alpha = d\sum \omega^s v_s = \sum_s d\omega^s v_s \mp \sum \omega^s \wedge dv_s$$

$$= \sum_{s,u} \omega^u \wedge \omega^s_u v_s - \sum \omega^{\cancel{s}\,u} \wedge \omega^{\cancel{u}\,s}_{\cancel{s}\,u} v_{\cancel{u}\,s} = 0$$

Theorem 6.5; $k < 0$

$k = -c^2$

$[e_1 \cdots e_n]$: an orthonormal frame $\quad \omega = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}$

(ω^j), $(\omega^i_j) = \Omega$, (K^i_j) : as usual

$\circledast \quad dF = F\hat{\Omega} \qquad \hat{\Omega} = \left(\begin{array}{c|c} 0 & c\,{}^t\omega \\ \hline c\omega & \Omega \end{array}\right)$

$K^i_j = -c^2 \omega^i \wedge \omega^j \Rightarrow \underline{d\hat{\Omega} + \hat{\Omega} \wedge \hat{\Omega} = 0}$

$F(p_0) = \mathrm{id}$.

$\exists F : U \xrightarrow{C^\infty} \cancel{M_{n+1}(\mathbb{R})}$ $SO_+(n,1)$ with $\circledast$

$$O(n,1) = \{ A \in M_{n+1}(\mathbb{R}) \,;\, \langle Ax, Ay \rangle_L = \langle x, y \rangle_L \}$$

Lorentzian group

$$= \{ A \in M_{n+1}(\mathbb{R}) \,;\, {}^tAYA = Y \} \qquad Y = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$SO(n,1) = \{ A \in O(n,1) \,;\, \det A = +1 \}$$

$$SO_+(n,1) = \{ A = (a_{ij}) \in SO(n,1) \,;\, a_{00} > 0 \}$$

$$(|a_{00}| \geqq 1)$$

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega} \qquad \widetilde{\Omega}Y + Y{}^t\widetilde{\Omega} = O \qquad \mathcal{F}(p_0) = \mathrm{id}$$

$$\Rightarrow \quad \mathcal{F} : U \to SO_+(n,1) \qquad \lrcorner$$

$$[\,v_0 \;\cdots\; v_n\,] \qquad f := \frac{1}{c} v_0 \text{ is a desired map.}$$

The fundamental theorem for surfaces

- $f\colon \mathbb{R}^2 \supset U \to \mathbb{R}^3$: an immserion
- $ds^2 = \langle df, df \rangle$: the first fundamental form, which gives a Riemannian metric on U.
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: an orthornomal frame on (U, ds^2)
- (ω^1, ω^2): the dual to $[\boldsymbol{e}_j]$
- Ω, K: the connection form and curvatrure form:

$$\Omega = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & -d\mu \\ d\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \omega^1 \wedge \omega^2$$

$$d\Omega + \Omega \wedge \Omega = \begin{pmatrix} 0 & -d\mu \\ d\mu & 0 \end{pmatrix}$$

$$\mu \wedge \mu = 0 \qquad + \cancel{\begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}}$$

The fundamental theorem for surfaces

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$\in \mathbb{R}^3$ $\qquad\qquad \nu$

$$\boldsymbol{v}_1 := df(\boldsymbol{e}_1), \qquad \boldsymbol{v}_2 := df(\boldsymbol{e}_2), \qquad \boldsymbol{v}_3 := \boldsymbol{v}_1 \times \boldsymbol{v}_2$$

- $\boxed{h^j := -\langle d\boldsymbol{v}_3, \boldsymbol{v}_j \rangle}$ $(j = 1, 2)$: the second fundamental form.

The fundamental theorem for surfaces

- $\mathcal{F} := (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \colon U \to \mathrm{SO}(3)$: the adapted frame

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & -\mu & -h^1 \\ \mu & 0 & -h^2 \\ h^1 & h^2 & 0 \end{pmatrix}.$$

$\boxed{\text{Ex 2}}$

$dv_1 = \cdots$

$dv_2 = \cdots$

$dv_3 = \cdots$

$\langle dv_1, v_1 \rangle$

$\langle dv_1, v_2 \rangle$

!

Exercise 6-1

Problem (Ex. 6-1)

Prove Theorem 6.5 for $k > 0$. ✓

Exercise 6-2

Problem (Ex. 6-2)

Prove Lemma 6.6.