Advanced Topics in Geometry F (MTH.B502)

Local uniqueness of space forms

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Isometry

Definition (Def. 6.2)

A C^∞-map $f\colon M \to N$ between Riemannian manifolds (M, g) and (N, h) is called a <u>local isometry</u> if $\dim M = \dim N$ and $f^*h = g$ hold, that is,

$$f^*h(X, Y) := h(df(X), df(Y)) = g(X, Y)$$

holds for X, $Y \in T_pM$ and $p \in M$.

- A local isometry is an immersion (Lemma 6.3)

Isometry

Corollary

A smooth map $f\colon (M,g) \to (N,h)$ is a local isometry if and only if for each $p \in M$,

$$[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] := [df(\boldsymbol{e}_1), \dots, df(\boldsymbol{e}_n)]$$

is an orthonormal frame for some orthonormal frame $[\boldsymbol{e}_j]$ on a neighborhood of p.

Local Uniqueness Theorem

Theorem (Thm. 6.5)

Let $U \subset \mathbb{R}^n$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (U, g) is constant k, there exists a local isometry $f\colon U \to N^n(k)$, where

$$N^n(k) = \begin{cases} S^n(k) & (k > 0) \\ \mathbb{R}^n & (k = 0) \\ H^n(k) & (k < 0). \end{cases}$$

Theorem 6.5; $k = 0$

Theorem 6.5; $k < 0$

The fundamental theorem for surfaces

- $f\colon \mathbb{R}^2 \supset U \to \mathbb{R}^3$: an immserion
- $ds^2 = \langle df, df \rangle$: the first fundamental form, which gives a Riemannian metric on U.
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: an orthornomal frame on (U, ds^2)
- (ω^1, ω^2): the dual to $[\boldsymbol{e}_j]$
- Ω, K: the connection form and curvatrure form:

$$\Omega = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & -d\mu \\ d\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \omega^1 \wedge \omega^2$$

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$$\boldsymbol{v}_1 := df(\boldsymbol{e}_1), \qquad \boldsymbol{v}_2 := df(\boldsymbol{e}_2), \qquad \boldsymbol{v}_3 := \boldsymbol{v}_1 \times \boldsymbol{v}_2$$

- $h^j := -\langle d\boldsymbol{v}_3, \boldsymbol{v}_j \rangle$ $(j = 1, 2)$: the second fundamental form.

The fundamental theorem for surfaces

- $\mathcal{F} := (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)\colon U \to \mathrm{SO}(3)$: the adapted frame

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & -\mu & -h^1 \\ \mu & 0 & -h^2 \\ h^1 & h^2 & 0 \end{pmatrix}.$$

Exercise 6-1

Problem (Ex. 6-1)

Prove Theorem 6.5 for $k > 0$.

Exercise 6-2

Problem (Ex. 6-2)

Prove Lemma 6.6.