Advanced Topics in Geometry F (MTH.B502)

Fundamental Theorem for surfaces

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Exercise 6-1

Problem (Ex. 6-1)

Prove Theorem 6.5 for $k > 0$.

Theorem (Thm. 6.5)

local uniqueness for k : const.

Let $U \subset \mathbb{R}^n$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (U, g) is constant k, there exists a local isometry $f : U \to N^n(k)$, where

$$N^n(k) = \begin{cases} S^n(k) & (k > 0) \\ \mathbb{R}^n & (k = 0) \\ H^n(k) & (k < 0). \end{cases}$$

skew symm.

$$\tilde{\Omega} = \left.\begin{pmatrix} 0 & -c\,{}^t\omega \\ c\omega & \underbrace{n\begin{pmatrix} \Omega \end{pmatrix}}_{n} \end{pmatrix}\right\} n+1$$

$(\circledast)$ $dF = F\tilde{\Omega}$; $F(p_0) = id$

sect. curvature $= k = c^2$

$\Rightarrow$ $(\circledast)$: integrable

$\Rightarrow$ $\exists F = (v_0 \cdots v_n) : U \longrightarrow SO(n+1)$

$(\bigstar)$ $x = \frac{1}{c} v_0 : U \to S^n(c^2) \subset \mathbb{R}^{n+1}$

is the desired one.

· $k = c^2$

$e_1 \cdots e_n$: orthonormal frame of (U, g)

· $\omega = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}$: the dual

· $\Omega = (\omega^i_j)$: conn. form

Exercise 6-2

$f : U \longrightarrow \mathbb{R}^3$ an immersion

Problem (Ex. 6-2)

Prove Lemma 6.6

ds^2 : the first fundamental form

- $ds^2(X\ Y) = \langle df(X), df(Y) \rangle$
- $[e_1\ \ e_2]$: orthonormal

Lemma (Lem. 6.6)

tangent $\longrightarrow v_j = df(e_j)$ $\quad v_3 = v_1 \times v_2$

unit normal

$$dv_1 = -\mu v_2 + h^1 v_3,$$
$$dv_2 = \mu v_1 + h^2 v_3,$$
$$dv_3 = -h^1 v_1 - h^2 v_2,$$

in other words,

$\mathcal{F} = [v_1, v_2, v_3]$

Gauss–Weingarten

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & \mu & -h^1 \\ -\mu & 0 & -h^2 \\ h^1 & h^2 & 0 \end{pmatrix}.$$

$[e_1, e_2]$ orthonormal $\quad [v_1, v_2] = [df(e_1), df(e_2)]$

$v_3 = v_1 \times v_2$

2nd fundamental form

- $h^j = -\langle dv_3, v_j \rangle \qquad h = h^1 e_1 + h^2 e_2$

$(dx = \omega^1 e_1 + \omega^2 e_2)$

$$df = \omega^1 v_1 + \omega^2 v_2 \qquad (\omega^1, \omega^2) : \text{dual to } (e_i)$$

$$dv_1 = \cancel{v_1} + \mu v_2 + h^1 v_3$$

$\langle dv_1, v_3 \rangle = -\langle v_1, dv_3 \rangle$

$$\langle dv_1, v_1 \rangle = \frac{1}{2} d\langle v_1, v_1 \rangle = 0$$

$\mu = \omega^1_2$ the connection form (later)

$$dv_1 = -\mu v_2 + h^1 v_3$$
$$dv_2 = \mu v_1 + h^2 v_3$$

$$dw^1 = \sum w^s \wedge w_s^1$$
$$= w^2 \wedge w_2^1$$
$$dw^2 = w^1 \wedge w_1^2$$
$$= - w^1 \wedge w_2^1$$

$$df = w^1 v_1 + w^2 v_2$$
$$0 = dw^1 \cdot v_1 + dw^2 v_2 - w^1 \wedge dv_1 - w^2 \wedge dv_2$$
$$= \left(\underline{w^2 \wedge w_2^1 - w^2 \wedge \mu} \right) v_1$$
$$+ \left(\underline{-w^1 \wedge w_2^1 + w^1 \wedge \mu} \right) v_2$$
$$- \boxed{\left(w^1 \wedge h^1 + w^2 \wedge h^2 \right)} v_3$$

$$\begin{cases} w^2 \wedge (w_2^1 - \mu) = 0 \\ w^1 \wedge (w_2^1 - \mu) = 0 \end{cases} \quad \therefore \ w_2^1 = \mu$$

Exercise 6-2

- $f\colon U \to \mathbb{R}^3$, $ds^2 = f^* \langle \ , \ \rangle$
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: an orthonormal frame of (U, ds^2); $\mu = \omega_2^1$: the connection form
- $\boldsymbol{v}_j = df(\boldsymbol{e}_j)$ $(j = 1, 2)$, $\boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2$
- $h^j = -\langle d\boldsymbol{v}_3, \boldsymbol{v}_j \rangle$.

$\Rightarrow$

$$\begin{aligned} d\boldsymbol{v}_1 &= -\mu \boldsymbol{v}_2 + h^1 \boldsymbol{v}_3, \\ d\boldsymbol{v}_2 &= \mu \boldsymbol{v}_1 + h^2 \boldsymbol{v}_3, \\ d\boldsymbol{v}_3 &= -h^1 \boldsymbol{v}_1 - h^2 \boldsymbol{v}_2, \end{aligned}$$