Advanced Topics in Geometry F (MTH.B502)

Fundamental Theorem for surfaces

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Exercise 6-1

Problem (Ex. 6-1)

Prove Theorem 6.5 for $k > 0$.

Theorem (Thm. 6.5)

Let $U \subset \mathbb{R}^n$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (U, g) is constant k, there exists a local isometry $f\colon U \to N^n(k)$, where

$$N^n(k) = \begin{cases} S^n(k) & (k > 0) \\ \mathbb{R}^n & (k = 0) \\ H^n(k) & (k < 0). \end{cases}$$

Exercise 6-2

Problem (Ex. 6-2)

Prove Lemma 6.6

Lemma (Lem. 6.6)

$$
\begin{aligned}
d\boldsymbol{v}_1 &= -\mu \boldsymbol{v}_2 + h^1 \boldsymbol{v}_3, \\
d\boldsymbol{v}_2 &= \mu \boldsymbol{v}_1 + h^2 \boldsymbol{v}_3, \\
d\boldsymbol{v}_3 &= -h^1 \boldsymbol{v}_1 - h^2 \boldsymbol{v}_2,
\end{aligned}
$$

in other words,

$$
d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & -\mu & -h^1 \\ \mu & 0 & -h^2 \\ h^1 & h^2 & 0 \end{pmatrix}.
$$

Correction: $\mu \to -\mu$

Exercise 6-2

- $f\colon U \to \mathbb{R}^3$, $ds^2 = f^* \langle\ ,\ \rangle$
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: an orthonormal frame of (U, ds^2); $\mu = \omega_2^1$: the connection form
- $\boldsymbol{v}_j = df(\boldsymbol{e}_j)$ $(j = 1, 2)$, $\boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2$
- $h^j = -\langle d\boldsymbol{v}_3, \boldsymbol{v}_j \rangle$.

$\Rightarrow$

$$\begin{aligned} d\boldsymbol{v}_1 &= -\mu \boldsymbol{v}_2 + h^1 \boldsymbol{v}_3, \\ d\boldsymbol{v}_2 &= \mu \boldsymbol{v}_1 + h^2 \boldsymbol{v}_3, \\ d\boldsymbol{v}_3 &= -h^1 \boldsymbol{v}_1 - h^2 \boldsymbol{v}_2, \end{aligned}$$