Advanced Topics in Geometry F (MTH.B502)

Fundamental Theorem for surfaces

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Surfaces in 3-manifolds

(N^3, g): a Riemannian 3-manifold.

- $f\colon M^2 \to N^3$: an immersion. (M^2, ds^2) : Riemannian mfd
- $ds^2(X,Y) := g(df(X), df(Y))$: the first fundamental form
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: an orthonormal frame on $(U \subset M^2, ds^2)$
- $[\boldsymbol{v}_1, \boldsymbol{v}_2] := [df(\boldsymbol{e}_1), df(\boldsymbol{e}_2)]$
- $\boldsymbol{v}_3$: the unit normal vector field

the second f.f

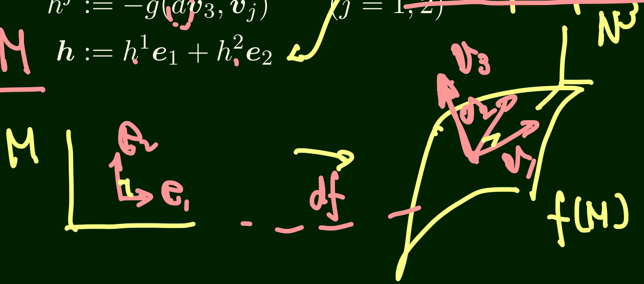
the shape operator

linear.

$$h^j := -g(d\boldsymbol{v}_3, \boldsymbol{v}_j) \qquad (j = 1, 2)$$

$h : T_pM \to T_pM$

$$\boldsymbol{h} := h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$$



The second fundamental forms

normal

$$h^j := -g(d\boldsymbol{v}_3, \boldsymbol{v}_j) \qquad (j = 1, 2)$$ 1 form

$$h_i^j = h^j(\boldsymbol{e}_i)$$

$$\boldsymbol{h} := h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$$

$$\begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}$$

Lemma (Lem. 7.1)

$h_2^1 = h_1^2$.

$$df = \omega^1 v_1 + \omega^2 v_2$$

$$0 = d\,df = \cdots v_1 + \cdots v_2 - \underline{\underline{(\omega^1 \wedge h^1 + \omega^2 \wedge h^2)}}\, v_3$$

$$\omega^1 \wedge h^1 + \omega^2 \wedge h^2 = 0$$

$$\Leftrightarrow \quad (\omega^1 \wedge h^1)(\theta_1, \theta_2) + (\omega^2 \wedge h^2)(\theta_1, \theta_2) = 0$$

$$\omega^1(\theta_1)\, h^1(\theta_2) - \cancel{\omega^1(\theta_2)\, h^1(\theta_1)}$$

$$+ \cancel{\omega^2(\theta_1)\, h^2(\theta_2)} - \omega^2(\theta_2)\, h^2(\theta_1) = 0$$

$$\omega^i(\theta_j) = \delta^i_j$$

$$\Leftrightarrow \quad h^1_2 - h^2_1 = 0$$

The second fundamental forms

$$h^j := -g(d\boldsymbol{v}_3, \boldsymbol{v}_j) \qquad (j = 1, 2)$$

$$h_i^j = h^j(\boldsymbol{e}_i)$$

$$\boldsymbol{h} := h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$$

$$\begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}$$

Definition (Def. 7.2)

外的曲率

exterior curvature

det

$$K_{\mathrm{ext}} := \underbrace{h_1^1 h_2^2 - h_2^1 h_1^2}_{} = h^1 \wedge h^2(\boldsymbol{e}_1, \boldsymbol{e}_2),$$

$$H = \frac{h_{11} + h_{22}}{2}.$$

the mean curvature

h_1^1 $\qquad$ h_2^2

the hyperbolic sp

$$N^3 = N^3(k_0) := \begin{cases} H^3(k_0) & (k_0 < 0), \quad SO(3,1) \\ \mathbb{R}^3 & (k_0 = 0), \quad SO(3) \ltimes \mathbb{R}^3 \\ S^3(k_0) \subset \mathbb{R}^4 & (k_0 > 0). \quad SO(4) \end{cases}$$

the sphere

(simply connected) space forms

空間形

(N^3, g) : Riem. mfd. connected (complete / compact of dist) Riem. mfd of const sec curv

$G := \mathrm{Isom}_0(N^3, g) :=$ the isometry group

$\parallel$

a Lie group

$\boxed{\dim G \leqq 6}$ "=" $\Leftrightarrow N^3 = N^3(k_0)$

曲面論の基本定理.

Theorem (The fundamental theorem for surfaces, Thm. 7.4)

- $U \subset \mathbb{R}^2$: *a simply connected domain*
- ds^2: *a Riemannian metric on* U. (1st ff)
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: *an orthonormal frame;* $\mu = \omega^1_2$: *the connection form.*
- k: *the sectional curvature.*
- h^1, h^2: *one forms on* U; $K_{\text{ext}} = h^1 \wedge h^2(\boldsymbol{e}_1, \boldsymbol{e}_2)$ ext. curvature

2nd f.f.

Assume

1. $k = K_{\text{ext}} + k_0$.
2. $dh^1 = h^2 \wedge \mu$, $dh^2 = -h^1 \wedge \mu$.

Integrability

$\Rightarrow \exists f : U \to N^3(k_0)$ *with* ds^2 *and* $\boldsymbol{h} = h^1 \boldsymbol{e}_1 + h^2 \boldsymbol{e}_2$ *as the fundamental forms.*

Surfaces in space Forms

- $f\colon U \to N^3(k_0)$: an immersion
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: an orthonormal frame; $\mu = \omega_2^1$: the connection form.
- $\boldsymbol{h} = h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$: the second fundamental form.
- $\mathcal{F} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$: an adapted frame.

$\boxed{k_0 = 0}$ $\mathbb{R}^3$ $v_j = df(e_j)$ $v_3 = v_1 \times v_2$

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & \mu & -h^1 \\ -\mu & 0 & -h^2 \\ h^1 & h^2 & 0 \end{pmatrix}$$

Ex. 6-2

$d\omega_2^1$ $=$ $d\mu = k\,\omega^1\wedge\omega^2$

integrability: $d\widetilde{\Omega} + \widetilde{\Omega}\wedge\widetilde{\Omega} = 0$

- $d\mu - h^1\wedge h^2 = 0$ $\quad K_{ext}\,\omega^1\wedge\omega^2$ Gauss
- $dh^1 = h^2\wedge\mu, \quad dh^2 = -h^1\wedge\mu$ Codazzi

Surfaces in space Forms

- $f\colon U \to N^3(k_0)$: an immersion
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v_0

$k_0 = -c^2 < 0$

cf

$K = K_{ext} + k_0$

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & c\omega^1 & c\omega^2 & 0 \\ c\omega^1 & 0 & \mu & -h^1 \\ c\omega^2 & -\mu & 0 & -h^2 \\ 0 & h^1 & h^2 & 0 \end{pmatrix}.$$

Integrability

$d\mu + c^2\omega^1\wedge\omega^2 - h^1\wedge h^2 = 0$

$dh^1 = h^2\wedge\mu, \quad dh^2 = -h^1\wedge\mu$

Codazzi

Same as the case $k_0 = 0$

Application

Lawson Correspondence. $H = \frac{1}{2}(h^1_1 + h^2_2)$

Theorem (Thm. 7.6)

Let $f\colon U \to N^3(k_0)$ be an immersion of constant mean curvature H defined on a simply-connected domain $U \subset \mathbb{R}^2$. Then there exists an immersion $f_{\tilde{k}_0}\colon U \to N^3(\tilde{k}_0)$ of constant mean curvature $H + t$ sharing the first fundamental form with f, where $\tilde{k}_0 = \cancel{k_0} - t^2 - \cancel{2H}t$.

$$\hat{h}^1 := h^1 + t\omega^1 \qquad \hat{h}^2 := h^2 + t\omega^2$$

$$d\hat{h}^1 = dh^1 + t\,d\omega^1 = h^2\wedge\mu + t\,\omega^2\wedge\mu$$

$$= (h^2 + t\omega^2)\wedge\mu = \hat{h}^2\wedge\mu \quad \text{etc}$$

$(\hat{h}^1, \hat{h}^2)$ satisfies the codazzi eq.

$$\tilde{h}^1 \wedge \tilde{h}^2 = (h^1 + t\omega^1) \wedge (h^2 + t\omega^2)$$

$$= h^1 \wedge h^2 + t\left(h^1 \wedge \omega^2 + \cancel{t}\,\omega^1 \wedge h^2\right)$$

$$+ t^2 \omega^1 \wedge \omega^2$$

(e_1, e_2)

$$\hat{K}_{ext} = K_{ext} + 2tH + t^2$$

$$\cancel{K} - \hat{k}_0 = \cancel{K} - k_0 + 2tH + t^2$$

Setting $\hat{k}_0 = k_0 - 2tH - t^2$

$(ds^2, \tilde{h}^1, \tilde{h}^2)$ satisfies the integrability in $N^3(\hat{k}_0)$

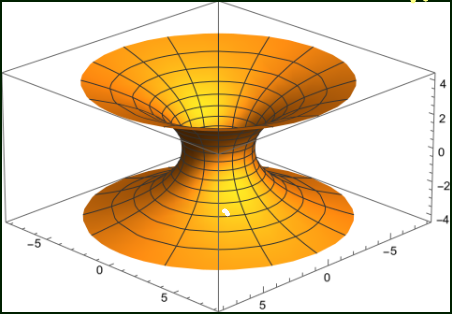
Example $k_0 = 0$ $H=0$ $k=-1$

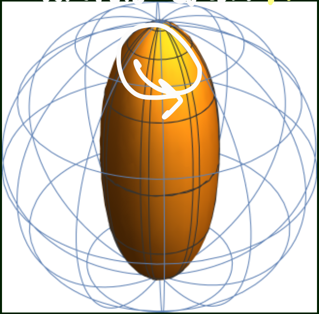
Let $f\colon U \to \mathbb{R}^3$ be a minimal surface (that is, with zero mean curvature). Then there exists $f_1\colon U \to H^3(-1)$ of constant mean curvature 1 with the same first fundamental form as f.

$H=0$ in $\mathbb{R}^3$	$H=1$ in $H^3(-1)$ f.
Weierstrass repr. ← complex analytic data	Bryant's formula 1986 ?

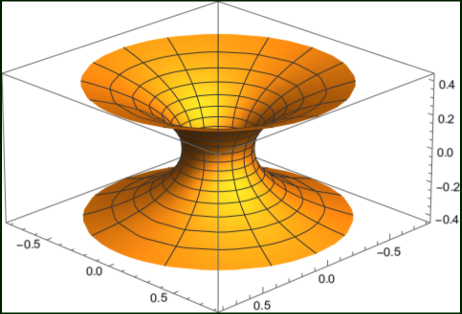
Example

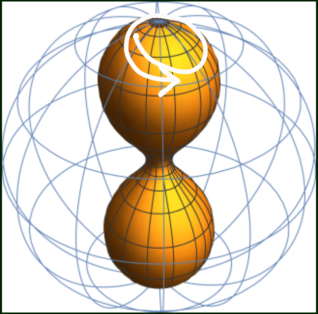
catenoid $\mathbb{R}^3$ catenoid cousin $H^3(-1)$



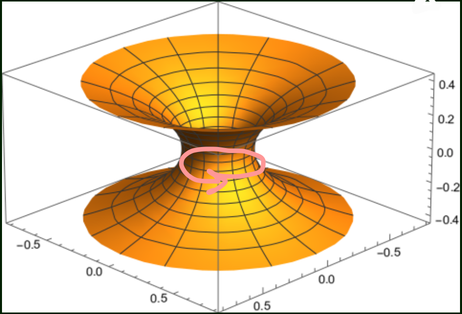


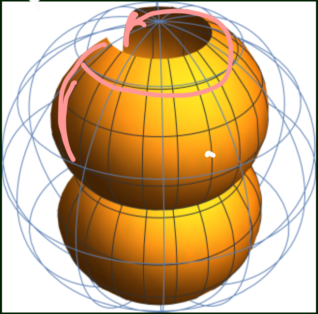
Poincaré model


0.4
0.2
0.0
−0.2
−0.4
−0.5
0.0
0.5
−0.5
0.0
0.5



Example





Lawson correspondence : local correspondence,

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