Advanced Topics in Geometry F (MTH.B502)

Fundamental Theorem for surfaces

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Surfaces in 3-manifolds

(N^3, g): a Riemannian 3-manifold.

- $f\colon M^2 \to N^3$: an immersion.
- $ds^2(X, Y) := g(df(X), df(Y))$: the first fundamental form
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: an orthonormal frame on $(U \subset M^2, ds^2)$
- $[\boldsymbol{v}_1, \boldsymbol{v}_2] := [df(\boldsymbol{e}_1), df(\boldsymbol{e}_2)]$
- $\boldsymbol{v}_3$: the unit normal vector field

$$h^j := -g(d\boldsymbol{v}_3, \boldsymbol{v}_j) \qquad (j = 1, 2)$$
$$\boldsymbol{h} := h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$$

The second fundamental forms

$$h^j := -g(d\boldsymbol{v}_3, \boldsymbol{v}_j) \qquad (j = 1, 2)$$
$$h_i^j = h^j(\boldsymbol{e}_i)$$
$$\boldsymbol{h} := h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$$

Lemma (Lem. 7.1)

$h_2^1 = h_1^2$.

The second fundamental forms

$$
\begin{aligned}
h^j &:= -g(d\boldsymbol{v}_3, \boldsymbol{v}_j) \qquad (j = 1, 2) \\
h^j_i &= h^j(\boldsymbol{e}_i) \\
\boldsymbol{h} &:= h^1 \boldsymbol{e}_1 + h^2 \boldsymbol{e}_2
\end{aligned}
$$

Definition (Def. 7.2)

$$
\begin{aligned}
K_{\text{ext}} &:= h^1_1 h^2_2 - h^1_2 h^2_1 = h^1 \wedge h^2(\boldsymbol{e}_1, \boldsymbol{e}_2), \\
H &:= \frac{h_{11} + h_{22}}{2}.
\end{aligned}
$$

The Fundamental Theorem

$$N^3 = N^3(k_0) := \begin{cases} H^3(k_0) & (k_0 < 0), \\ \mathbb{R}^3 & (k_0 = 0), \\ S^3(k_0) & (k_0 > 0). \end{cases}$$

Surfaces in space Forms

Theorem (The fundamental theorem for surfaces, Thm. 7.4)

- $U \subset \mathbb{R}^2$:*a simply connected domain*
- ds^2: *a Riemannian metric on* U.
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: *an orthonormal frame;* $\mu = \omega^1_2$: *the connection form.*
- k: *the sectional curvature.*
- h^1, h^2: *one forms on* U; $K_{\mathrm{ext}} = h^1 \wedge h^2(\boldsymbol{e}_1, \boldsymbol{e}_2)$

Assume

1. $k = K_{\mathrm{ext}} + k_0$.
2. $dh^1 = h^2 \wedge \mu$, $dh^2 = -h^1 \wedge \mu$.

$\Rightarrow \exists f : U \to N^3(k_0)$ *with* ds^2 *and* $\boldsymbol{h} = h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$ *as the fundamental forms.*

Surfaces in space Forms

- $f\colon U \to N^3(k_0)$: an immersion
- $[\boldsymbol{e}_1, \boldsymbol{e}_2]$: an orthonormal frame; $\mu = \omega_2^1$: the connection form.
- $\boldsymbol{h} = h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$: the second fundamental form.
- $\mathcal{F} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$: an adapted frame.

$k_0 = 0$

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & \mu & -h^1 \\ -\mu & 0 & -h^2 \\ h^1 & h^2 & 0 \end{pmatrix}$$

Surfaces in space Forms

- $f \colon U \to N^3(k_0)$: an immersion
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$k_0 = -c^2 < 0$

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & c\omega^1 & c\omega^2 & 0 \\ c\omega^1 & 0 & \mu & -h^1 \\ c\omega^2 & -\mu & 0 & -h^2 \\ 0 & h^1 & h^2 & 0 \end{pmatrix}.$$

Application

Theorem (Thm. 7.6)

Let $f : U \to N^3(k_0)$ *be an immersion of constant mean curvature* H *defined on a simply-connected domain* $U \subset \mathbb{R}^2$. *Then there exists an immersion* $f_{\tilde{k}_0} : U \to N^3(\tilde{k}_0)$ *of constant mean curvature* $H + t$ *sharing the first fundamental form with* f, *where* $\tilde{k}_0 = k_0 - t^2 - 2Ht$.

Example

Example

Let $f\colon U \to \mathbb{R}^3$ be a minimal surface (that is, with zero mean curvature). Then there exists $f_1\colon U \to H^3(-1)$ of constant mean curvature 1 with the same first fundamental form as f_1.