1 Riemannian metrics

In this section, we deal with *differentiable manifolds*. For a student not familiar to manifold theory, it is sufficient to imagine a domain of the Euclidean space, since our purpose is local theory of Riemannian manifolds.

Let M be an n-dimensional *differentiable manifold*, and denote by $\mathcal{F}(M)$ the set of *differentiable functions* on M. Take a coordinate neighborhood $(U; u^1, \dots, u^n)$ on M.

Tangent spaces For a fixed point $p \in M$, a linear map $X \colon \mathcal{F}(M) \to \mathbb{R}$ satisfying the Leibniz rule $X(fg) = f(p)Xg + g(p)Xf$ is called a *tangent vector* of M at p. Taking a local coordinate system $(U; (x^1, \dots, x^n))$ containing p, the maps

$$\left(\frac{\partial}{\partial x^j}\right)_p : \mathcal{F}(M) \ni f \longmapsto \left(\frac{\partial}{\partial x^j}\right)_p f = \frac{\partial f \circ \varphi^{-1}}{\partial x^j}(\varphi(p)) \in \mathbb{R}$$

are tangent vectors at p. The set of tangent vectors of M at p is called the *tangent space* of M at p, and denoted by T_pM . The tangent space is an n $(= \dim M)$-dimensional vector space spanned by

$$\left[\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^n}\right)_p\right]. \tag{1.1}$$

Take another local coordinate system $(V, (y^1, \dots, y^n))$. Then the coordinate change $(x^j) \mapsto (y^l)$ is a local diffeomorphism between domains in $\mathbb{R}^n$. Using this, we can write

$$\left(\frac{\partial}{\partial x^j}\right)_p = \sum_{k=1}^n \frac{\partial y^k}{\partial x^j}(p)\left(\frac{\partial}{\partial y^k}\right)_p. \tag{1.2}$$

On the other hand, if $X \in T_pM$ is written as

$$X = \sum_{j=1}^n X^j \left(\frac{\partial}{\partial x^j}\right)_p = \sum_{k=1}^n \tilde{X}^k \left(\frac{\partial}{\partial y^k}\right)_p,$$

it holds that

$$\tilde{X}^k = \sum_{j=1}^n \frac{\partial y^k}{\partial x^j}(p) X^j. \tag{1.3}$$

Cotangent space. The *dual space* of the vector space T_pM is called the *cotangent space* at p, and denoted by T_p^*M. In other words,

$$T_p^*M = \{\alpha \colon T_pM \to \mathbb{R}; \alpha \text{ is a linear map}\}.$$

We denote the dual basis of (1.1) by

$$\left[(dx^1)_p, \dots, (dx^n)_p\right]. \tag{1.4}$$

That is, $(dx^j)_p \in T_p^*M$ is defined as

$$(dx^j)_p\left(\left(\frac{\partial}{\partial x^k}\right)_p\right) = \delta^j_k.$$

By (1.2), we have

$$(dy^k)_p = \sum_{j=1}^n \frac{\partial y^k}{\partial x^j}(p)(dx^j)_p. \tag{1.5}$$

14. June, 2022. Revised: 19. July, 2022

1.1 The tangent bundle and vector fields.

We denote the direct union of tangent spaces by

$$TM := \bigcup_{p\in M} T_pM,$$

and called the *tangent bundle* of M. Define a map

$$\tilde{\varphi}\colon \pi^{-1}(U) \ni X = \sum_{j=1}^{n} X^j \left(\frac{\partial}{\partial x^j}\right)_p \longmapsto \bigl(\varphi(p), X^1, \dots, X^n\bigr) \in \varphi(U) \times \mathbb{R}^n$$

for each local coordinate system $(U, \varphi = (x^1, \dots, x^n))$, where $\pi\colon TM \to M$ is the canonical projection. By these, TM can be considered as a differentiable manifold of dimension $2n$.

A *vector field* on M is a map $X\colon M \to TM$ satisfying $\pi \circ X = \mathrm{id}_M$, where id_M is the identity map on M. In other words, X is a "smooth" correspondence to each point p to a tangent vector $X_p \in T_pM$. Using local coordinate system, one can write

$$X = \sum_{j=1}^{n} X^j(x^1, \dots, x^n)\frac{\partial}{\partial x^j} \qquad (\ X^j(x^1, \dots, x^n) \text{ are differentiable functions in } (x^k)), \tag{1.6}$$

where $\partial/\partial x^j$ is a local vector field given by $p \mapsto (\partial/\partial x^j)_p$.

Vector bundles induced from the tangent bundle. The *cotangent bundle* is the union of cotangent spaces $T^*M = \cup_{p\in M} T^*_pM$ with appropriate structure of a $2n$-manifold. Taking tensor products of tangent and cotangent spaces, one can consider the tensor product of tangent and cotangent bundles, for example,

$$T^*M \otimes T^*M := \bigcup_{p\in M} T^*_pM \otimes T^*_pM.$$

In general, a triple (E, M, π) of differentiable manifolds E and M, and a differentiable map $\pi\colon E \to M$ is called a *vector bundle* on M if it satisfies the following:

- π is a surjection.
- For each $p \in M$, $E_p := \pi^{-1}(p)$ is endowed with a structure of N-dimensional vector space.
- There exists an open cover $\{U_\alpha\}$ on M and a family $\{\tilde{\varphi}_\alpha\}$ of diffeomorphisms $\tilde{\varphi}_\alpha\colon \pi^{-1}(U) \to U \times \mathbb{R}^N$ such that $\tilde{\varphi}_\alpha|_{E_p}\colon E_p \to \{p\} \times \mathbb{R}^N \simeq \mathbb{R}^N$ is a linear map for each p.

The tangent bundle TM, the cotangent bundle T^*M and the tensor product $T^*M \otimes T^*M$ are vector bundles on M.

A *section* of a vector bundle (E, M, π) (often denoted by E for simplicity) is a differentiable map $\xi\colon M \to E$ satisfying $\pi \circ \xi = \mathrm{id}_M$. The set of sections of a vector bundle E is denoted by $\Gamma(E)$. In particular, the set of vector fields is denoted by

$$\mathfrak{X}(M) = \Gamma(TM).$$

In general $\Gamma(E)$ is a structure of infinite dimensional vector space. Moreover, it is also considered as an $\mathcal{F}(M)$-module.

Riemannian metrics A *bilinear form* on M is a section of the vector bundle

$$S(T^*M\otimes T^*M)=\cup_{p\in M}S(T^*_pM\otimes T^*_pM),$$
$$S(T^*_pM\otimes T^*_pM)=\{\text{ (symmetric) bilinear forms on } T_pM\ \},$$

where

$$\begin{aligned}S(T^*_pM\otimes T^*_pM)&=\{Q\in T^*_pM\otimes T^*_pM\,;Q(X,Y)=Q(Y,X)\text{ for }X,Y\in T_pM\}\\&=\{Q\colon T_pM\times T_pM\to\mathbb{R};\text{bilinear, and}Q(X,Y)=Q(Y,X)\text{ for }X,Y\in T_pM\}\end{aligned}$$

Lemma 1.1. *A map $Q\colon p\mapsto Q_p$, where Q_p is a bilinear form on T_pM, is a smooth section of $S(T^*M\otimes T^*M)$ if and only if*

$$M\ni p\longmapsto Q_p(X_p,Y_p)\in\mathbb{R}$$

for each pair of smooth vector fields X, $Y\in\mathfrak{X}(M)$.

Definition 1.2. A Riemannian metric (resp. pseudo Riemannian metric) is a section $g\in\Gamma(S(T^*M\otimes T^*M))$ such that the quadratic form g_p on T_pM is positive definite inner product (resp. non-degenerate inner product) on T_pM. A pair (M,g) of a manifold M and a (pseudo) Riemannian metric g is called a *(pseudo) Riemannian manifold*.

Let (M,g) be a Riemannian manifold. Then, on a local coordinate system $(U,(x^j))$ on M, g is expressed as

$$g=\sum_{i,j=1}^n g_{ij}dx^i\otimes dx^j\qquad g_{ij}=g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right).$$

Moreover, each g_{ij} is a smooth function on U, and the matrix (g_{ij}) is a positive definite symmetric matrix. We often abbreviate $g(X,Y)$ by $\langle X,Y\rangle$.

Fact 1.3. *There exists a Riemannian metric on an arbitrary (paracompact) manifold.*

Example 1.4 (Euclidean spaces)**.** As a differentiable n-manifold, the tangent space $T_p\mathbb{R}^n$ of $\mathbb{R}^n$ can be identified with (the vector space) $\mathbb{R}^n$. More precisely, let $(x^1,\dots,x^n)$ be the canonical coordinate system of $\mathbb{R}^n$. Then

$$T_p\mathbb{R}^n\ni X=\sum X^j\left(\frac{\partial}{\partial x^j}\right)_p\leftrightarrow(X^1,\dots,X^n)\in\mathbb{R}^n$$

is the identification map of $T_p\mathbb{R}^n$ with $\mathbb{R}^n$.

Then the Euclidean inner product of $\mathbb{R}^n$ can be regarded as an inner product of $T_p\mathbb{R}^n$. Hence we obtain the *canonical Riemannian metric* g_0 on $\mathbb{R}^n$. Under the canonical coordinate system $(x^1,\dots,x^n)$, g_0 is written as

$$g_0=dx^1\otimes dx^1+dx^2\otimes dx^2+\dots+dx^n\otimes dx^n=(dx^1)^2+(dx^2)^2+\dots+(dx^n)^2.$$

Example 1.5 (Submanifolds)**.** Let N be a manifold, and $M\subset N$ a submanifold. Then for each $p\in M$, the tangent space T_pM is a linear subspace of T_pN. If a Riemannian metric g on N is given, the restriction of g on T_pM gives a Riemannian metric on M. Such a metric is called the metric on M induced by g, and denoted by $g|_M$.

Example 1.6 (Spheres)**.** Let $k=c^2$ be a positive number and set

$$S^n(k):=\left\{\boldsymbol{x}=(x^1,\dots,x^{n+1})\in\mathbb{R}^{n+1}\mid\langle x,x\rangle=\sum_{j=1}^{n+1}(x^j)^2=\frac{1}{k}\right\},$$

where $(x^1,\dots,x^{n+1})$ is the standard coordinate system on $\mathbb{R}^{n+1}$. Then one can easily see that $S^n(k)$ is a submanifold of $\mathbb{R}^{n+1}$, and hence, the canonical metric g_0 of $\mathbb{R}^{n+1}$ induces a Riemannian metric on $S^n(k)$. In particular, the metric on $S^n=S^n(1)$ induced from the Euclidean metric is called the *canonical metric on S^n*.

Example 1.7 (The Lorentz Minkowski space)**.** An inner product

$$\langle X, Y\rangle_L := -X^0Y^0 + X^1Y^1 + \dots X^nY^n$$
$$X = (X^0, X^1, \dots, X^n), \quad Y = (Y^0, Y^1, \dots, Y^n)$$

on $\mathbb{R}^{n+1}$ of signature $(n, 1)$ is called the *Minkowski inner product* , and the pair $\mathbb{R}^{n+1}_1 := (\mathbb{R}^{n+1}, \langle\ ,\ \rangle)$ the *Minkowski vector space*. A vector $X \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ is said to be *space-like*, *light-like* or *null*, and *time-like* , if $\langle X, X\rangle_L > 0$, $\langle X, X\rangle_L = 0$, and $\langle X, X\rangle_L < 0$, respectively.

In the same way as the Euclidean space case, the inner product $\langle\ ,\ \rangle_L$ induces the structure of pseudo Riemannian metric g_L on $\mathbb{R}^{n+1}$. We call the pseudo Riemannian manifold $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, g_L)$ the *Lorentz-Minkowski space*, or *Minkowski space*. The metric g_L is expressed as

$$g_L := -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n = -(dx^0)^2 + (dx^1)^2 + \dots + (dx^n)^2$$

under the canonical coordinate system $(x^0, \dots, x^n)$.

Example 1.8 (Hyperbolic spaces)**.** First, we claim (cf. Exercise 1-2)

> Let $\boldsymbol{v}$ be a time-like vector on the Minkowski vector space $\mathbb{R}^{n+1}_1$. Then its orthogonal complement $\{X \mid \langle X, \boldsymbol{v}\rangle_L = 0\}$ is an n-dimensional subspace of $\mathbb{R}^{n+1}_1$ consists of space-like vectors and $\mathbf{0}$.

For a negative constant $k = -c^2$ $(c \in \mathbb{R} \setminus \{0\})$, we set

$$H^n(-c^2) := \left\{\boldsymbol{x} = (x^0, \dots, x^n) \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle_L = -\frac{1}{c^2}, cx_0 > 0\right\}.$$

Then one can easily see that $H^n(-c^2)$ is a connected submanifold of $\mathbb{R}^{n+1}_1$.

For each point $\boldsymbol{x} \in H^n$, the tangent space of $H^n(-c^2)$ is expressed as

$$T_{\boldsymbol{x}}H^n(-c^2) = \{\boldsymbol{v} \in L^{n+1} \mid \langle \boldsymbol{x}, \boldsymbol{v}\rangle_L = 0\}.$$

In other words, the tangent space is an orthogonal complement of the position vector (which is time-like). Then $T_{\boldsymbol{x}}H^n(-c^2)$ consists of space-like vectors, that is, the restriction g_L to the tangent space of $H^n(-c^2)$ is positive definite. Hence it induces the Riemannian metric g_H. The Riemannian manifold $(H^n(-c^2), g_H)$ (resp. $(H^n = H^n(-1), g_H)$) is called the *hyperbolic space*.

Exercises

1-1 Let $U \subset \mathbb{R}^n$ be a domain and g a Riemannian metric on U. Show that

(1) There exists an n-tuple of vector fields $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ such that

$$g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (\text{otherwise}) \end{cases}.$$

(2) Take another n-tuple $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ satisfying $g(\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{ij}$. Then there exists a matrix-valued function

$$\Theta\colon U \to \mathrm{O}(n) \qquad [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta.$$

1-2 Let $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$ be the Minkowski vector space. Show that if $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ satisfies $\langle \boldsymbol{v}, \boldsymbol{v}\rangle_L = -1$, the orthogonal complement

$$\boldsymbol{v}^\perp := \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \,;\, \langle \boldsymbol{v}, \boldsymbol{x}\rangle_L = 0\}$$

is an n-dimensional space-like subspace of $\mathbb{R}^{n+1}_1$.