2 Riemannian Connection

2.1 Preliminal materials

Lie brackets Let M be an n-dimensional manifold and denote by $\mathcal{F}(M)$ and $\mathfrak{X}(M)$ the set of smooth function and the set of smooth vector fields on M, respectively. A vector field $X \in \mathfrak{X}(M)$ can be considered as a differential operator acting on $\mathcal{F}(M)$ as $(Xf)(\mathrm{P}) = X_{\mathrm{P}} f$. By definition it satisfies the Leibniz rule

$$X(fg) = f(Xg) + g(Xf) \qquad (X \in \mathfrak{X}(M), f, g \in \mathcal{F}(M)). \tag{2.1}$$

For two vector fields X, $Y \in \mathfrak{X}(M)$, set

$$[X,Y]\colon \mathcal{F}(M) \ni f \longmapsto X(Yf) - Y(Xf) \in \mathcal{F}(M). \tag{2.2}$$

Then $[X,Y]$ also satisfies the Leibnitz rule (2.1), and gives a vector field on M. The map

$$[\ ,\ ]\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X,Y) \mapsto [X,Y] \in \mathfrak{X}(M)$$

is called the *Lie bracket* on $\mathfrak{X}(M)$. One can easily show that the product $[\ ,\ ]$ is bilinear, skew symmetric and satisfies the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = \mathbf{0}, \tag{2.3}$$

that is, $(\mathfrak{X}(M), [\ ,\ ])$ is a *Lie algebra* (of infinite dimension). By the Leibniz rule, it holds that

$$[fX,Y] = f[X,Y] - (Yf)X, \quad [X,fY] = f[X,Y] + (Xf)Y \qquad (X,Y \in \mathfrak{X}(M), f \in \mathcal{F}(M)). \tag{2.4}$$

The Lie bracket can be considered a kind of "integrability condition" as follows:

Fact 2.1. *Let $(X_1,\dots,X_n)$ be an n-tuple of vector fields on n-dimensional manifolds, which is linearly independent in T_pM for each $p \in M$. Then existence of local coordinate system $(x^1,\dots,x^n)$ around p such that $\partial/\partial x^j = X_j$ $(j = 1,\dots,n)$ is equivalent to that $[X_j, X_k] = \mathbf{0}$ holds for all j, $k = 1,\dots,n$.*

Tensors. A section $\omega \in \Gamma(T^*M)$ of the cotangent bundle T^*M is called a *covariant* 1*-tensor* or a 1*-form*. A one form ω induces a linear map

$$\omega\colon \mathfrak{X}(M) \ni X \longmapsto \omega(X) \in \mathcal{F}(M), \qquad \text{where} \qquad \omega(X)(p) = \omega_p(X_p) \tag{2.5}$$

By definition, it holds that

$$\omega(fX) = f\omega(X) \qquad (f \in \mathcal{F}(M), X \in \mathfrak{X}(M)). \tag{2.6}$$

Lemma 2.2. *A linear map $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ is a 1-form if and only if (2.6) holds.*

Proof. The "only if" part is trivial by definition. Assume a linear map $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ satisfies (2.6). In fact, under a local coordinate system $(x^1,\dots,x^n)$ around $p \in M$,

$$\omega(X)(p) = \omega\left(\sum_{j=1}^n X^j \frac{\partial}{\partial x^j}\right)(p) = \sum_{j=1}^n X^j(p)\omega\left(\frac{\partial}{\partial x^j}\right)_p, \qquad \left(X = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}.\right)$$

holds. In other words, $\omega(X)(p)$ depend only on X_p. Hence ω induces a map $\omega_p\colon T_pM \to \mathbb{R}$. □

Similarly, a *covariant two tensor* $\alpha \in \Gamma(T^*M \otimes T^*M)$ induces a bilinear map $\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$. By the same reason as Lemma 2.2, we have

Lemma 2.3. *A bilinear map $\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$ is a $(0,2)$-tensor if and only if*

$$\alpha(fX,Y) = \alpha(X,fY) = f\alpha(X,Y) \qquad (f \in \mathcal{F}(M), X,Y \in \mathfrak{X}(M))$$

holds.

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The Exterior derivative. We denote the set of skew-symmetric covariant two tensors by

$$\wedge^2(M) := \left\{\omega \in \mathit{\Gamma}\bigl(T^*M \otimes T^*M\bigr)\,;\, \omega(X,Y) = -\omega(Y,X)\right\}.$$

An element $\omega \in \wedge^2(M)$ is called a *2-form* . Under such a context, the set of 1-forms and the set of smooth functions are denoted by

$$\wedge^1(M) := \mathit{\Gamma}(T^*M), \qquad \wedge^0 := \mathcal{F}(M).$$

The *exterior product* $\alpha \wedge \beta \in \wedge^2(M)$ of two 1-forms α, $\beta \in \wedge^1(M)$ is defined as

$$(\alpha\wedge\beta)(X,Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \tag{2.7}$$

Under a local coordinate system $(x^1,\dots,x^n)$, a one form α and a two form ω are expressed as

$$\alpha = \sum_{j=1}^{n} \alpha_j\, dx^j, \qquad \omega = \sum_{1\leqq i<j\leqq n} \omega_{ij}\, dx^i \wedge dx^j,$$

where α_j $(j = 1,\dots,n)$ and ω_{ij} $(1 \leqq i < j \leqq n)$ are smooth functions in $(x^1,\dots,x^n)$. By Lemma 2.3 and the property (2.4) of the Lie brackets, we have

Lemma 2.4. *For a function $f \in \mathcal{F}(M) = \wedge^0(M)$ and a 1-form $\alpha \in \wedge^1(M)$,*

$$\begin{aligned} df &\colon \mathfrak{X}(M) \ni X \mapsto df(X) = Xf \in \mathcal{F}(M),\\ d\alpha &\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \ni (X,Y) \mapsto X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) \in \mathcal{F}(M) \end{aligned}$$

are a 1-form and a 2-form, respectively.

Definition 2.5. For a function f and a one form α, df and $d\alpha$ are called the *exterior derivatives* of f and α, respectively.

2.2 The Riemannian connection.

Let (M,g) be an n-dimensional (pseudo) Riemannian manifold, and denote by $\langle\ ,\ \rangle$ the inner product induced by g.

Lemma 2.6. *A map $\flat\colon T_pM \ni X \mapsto X^\flat = \langle X,\cdot\rangle \in T_p^*M$ is a linear isomorphism.*

Proof. The linearity is trivial. Since $g = \langle\ ,\ \rangle$ is non-degenerate,

$$\operatorname{Ker}\flat = \{X \in T_pM\,;\, \langle X,Y\rangle = 0 \text{ for all } Y \in T_pM\} = \{\mathbf{0}\}.$$

The conclusion follows noticing that both T_pM and T_p^*M are n-dimensional. □

We denote by $\#\colon \alpha \mapsto \alpha^\#$ the inverse of $\flat$. Then $\#$ and $\flat$ induces an isomorphism between $\mathfrak{X}(M)$ and $\wedge^1(M)$.

Definition, existence and uniqueness.

Lemma 2.7. *There exists the unique bilinear map $\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \ni (X,Y) \mapsto \nabla_XY \in \mathfrak{X}(M)$ satisfying*

$$\nabla_XY - \nabla_YX = [X,Y], \qquad X\langle Y,Z\rangle = \langle\nabla_XY,Z\rangle + \langle X,\nabla_XZ\rangle \qquad (X,Y,Z\in\mathfrak{X}(M)) \tag{2.8}$$

Proof. Assume such a ∇ exists. Then for X, Y, $Z \in \mathfrak{X}(M)$,

$$\begin{aligned}\langle\nabla_X Y, Z\rangle &= X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_Z X + [X, Z]\rangle \\ &= X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - \langle Y, \nabla_Z X\rangle = X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle\nabla_Z Y, X\rangle \\ &= X\langle Y, Z\rangle - Z\langle Y, X\rangle - \langle Y, [X, Z]\rangle + \langle\nabla_Y Z, X\rangle + \langle[Z, Y], X\rangle \\ &= X\langle Y, Z\rangle - Z\langle Y, X\rangle - \langle Y, [X, Z]\rangle + \langle[Z, Y], X\rangle + Y\langle Z, X\rangle - \langle Z, \nabla_Y X\rangle \\ &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle Y, X\rangle - \langle Y, [X, Z]\rangle + \langle[Z, Y], X\rangle - \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle .\end{aligned}$$

Then

$$\begin{aligned}2\langle\nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle Y, X\rangle \\ &\qquad - \langle Y, [X, Z]\rangle + \langle[Z, Y], X\rangle - \langle Z, [Y, X]\rangle =: 2C(X, Y, Z).\end{aligned} \tag{2.9}$$

Hence, non-degeneracy of $\langle\ ,\ \rangle$ implies the uniqueness. Moreover, setting $\nabla_X Y := \bigl(C(X, Y, *)\bigr)^{\#}$, we have the existence. □

Definition 2.8. The map ∇ in Lemma 2.7 is called the *Riemannian connection* or the *Levi-Civita connection* of (M, g).

Lemma 2.9. *The Riemannian connection* ∇ *satisfies*

$$\nabla_{fX} Y = f\nabla_X Y, \qquad \nabla_X(fY) = (Xf)Y + f\nabla_X Y. \tag{2.10}$$

Proof. The conclusion follows from (2.4) and (2.9). □

Remark 2.10. A bilinear map $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying (2.10) is called a *linear connection* or an *affine connection*.

Remark 2.11. By Lemmas 2.9 and 2.2, $X \mapsto \nabla_X Y$ determines a one form.

Orthonormal frames.

Definition 2.12. Let $U \subset M$ be a domain of M. An n-tuple of vector fields $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ on U is called an *orthonormal frame* on U if $\langle\boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \delta_{ij}$. It is said to be *positive* if M is oriented and $\{\boldsymbol{e}_j\}$ is compatible to the orientation on M.

Exercise 1-1 assert that for each $p \in M$, there exists a neighborhood U of p which admits an orthonormal frame on U. Moreover, we have

Lemma 2.13. *Let* $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *be two orthonormal frames on* $U \subset M$. *Then there exists a smooth map*

$$\Theta\colon U \longrightarrow \mathrm{O}(n) \qquad \textit{such that} \qquad [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta. \tag{2.11}$$

Moreover, if $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *determines the common orientation,* Θ *is valued on* $\mathrm{SO}(n)$.

The map Θ in Lemma 2.13 is called a *gauge transformation*.

For an orthonormal frame $\{\boldsymbol{e}_j\}$ on U, we denote by $\{\omega^j\}_{j=1,\dots,n}$ the *dual frame* of $\{\boldsymbol{e}_j\}$, that is, $\omega^j \in \wedge^1(U)$ such that

$$\omega^j(\boldsymbol{e}_k) = \delta^j_k = \begin{cases} 1 & (j = k) \\ 0 & (\text{otherwise}). \end{cases}$$

In other words, $\omega^j(X) = \langle\boldsymbol{e}_j, X\rangle$.

Lemma 2.14. *Two orthonormal frames* $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *are related as* (2.11). *Then their duals* $\{\omega^j\}$ *and* $\{\lambda^j\}$ *satisfy*

$$\begin{pmatrix}\lambda^1\\ \vdots \\ \lambda^n\end{pmatrix} = \Theta \begin{pmatrix}\omega^1\\ \vdots \\ \omega^n\end{pmatrix}.$$

Proof.

$$\begin{pmatrix}\lambda^1\\ \vdots \\ \lambda^n\end{pmatrix}(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n) = \begin{pmatrix}\lambda^1\\ \vdots \\ \lambda^n\end{pmatrix}(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)\Theta = \Theta = \Theta\begin{pmatrix}\omega^1\\ \vdots \\ \omega^n\end{pmatrix}(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n). \qquad \square$$

Connection forms.

Definition 2.15. The *connection form* with respect to an orthonormal frame $\{\boldsymbol{e}_j\}$ is a $n\times n$-matrix valued one form Ω on U defined by

$$\Omega = \begin{pmatrix}\omega_1^1 & \omega_2^1 & \dots & \omega_n^1\\ \omega_1^2 & \omega_2^2 & \dots & \omega_n^2\\ \vdots & \vdots & \ddots & \vdots\\ \omega_1^n & \omega_2^n & \dots & \omega_n^n\end{pmatrix}, \qquad \omega_j^k := \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k\rangle \in \wedge^1(U).$$

By definition, we have $\nabla \boldsymbol{e}_j = \sum_{k=1}^n \omega_j^k \boldsymbol{e}_k$, that is, $\nabla[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n] = [\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]\Omega$.

Lemma 2.16. $\omega_j^k = -\omega_k^j$.

Proof. $\omega_j^k = \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k\rangle = d\langle \boldsymbol{e}_j, \boldsymbol{e}_k\rangle - \langle \boldsymbol{e}_j, \nabla\boldsymbol{e}_k\rangle = -\omega_k^j$. $\square$

Lemma 2.17. $d\omega^i = \sum_{l=1}^n \omega^l \wedge \omega_l^i$.

Proof.

$$\begin{aligned} d\omega^i(\boldsymbol{e}_j,\boldsymbol{e}_k) &= \boldsymbol{e}_j\omega^i(\boldsymbol{e}_k) - \boldsymbol{e}_k\omega^i(\boldsymbol{e}_j) - \omega^i([\boldsymbol{e}_j,\boldsymbol{e}_k]) = -\omega^i([\boldsymbol{e}_j,\boldsymbol{e}_k])\\ &= -\omega^i(\nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k - \nabla_{\boldsymbol{e}_k}\boldsymbol{e}_j) = -\left\langle \nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k - \nabla_{\boldsymbol{e}_k}\boldsymbol{e}_j, \boldsymbol{e}_i\right\rangle = -\omega_k^i(\boldsymbol{e}_j) + \omega_j^i(\boldsymbol{e}_k)\\ &= \sum_{l=1}^n\bigl(-\omega_l^i(\boldsymbol{e}_j)\omega^l(\boldsymbol{e}_k) + \omega_l^i(\boldsymbol{e}_k)\omega^l(\boldsymbol{e}_j)\bigr) = \sum_{l=1}^n \omega^l\wedge\omega_l^i(\boldsymbol{e}_j,\boldsymbol{e}_k). \qquad \square\end{aligned}$$

Exercises

2-1 Let $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ be two orthonormal frames on a domain U of a Riemannian n-manifold M, which are related as (2.11). Show that the connection forms Ω of $\{\boldsymbol{e}_j\}$ and Λ of $\{\boldsymbol{v}_j\}$ satisfy $\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta$.

2-2 Let $\mathbb{R}_1^3$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-c^2)$ the hyperbolic 2-space (i.e. the hyperbolic plane) as defined in Example 1.8. Verify that

$$(u,v)\mapsto\left(\frac{1}{c}\cosh cu, \frac{\cos v}{c}\sinh cu, \frac{\sin v}{c}\sinh cu\right)$$

gives a local coordinate system on $U := H^2(-c^2)\setminus\{(1/c,0,0)\}$, and

$$\boldsymbol{e}_1 := (\sinh cu, \cos v\cosh cu, \sin v\cosh cu), \qquad \boldsymbol{e}_2 := (0, -\sin v, \cos v)$$

forms a orthonormal frame on U.