

### 3 Curvature form as an integrability condition

#### 3.1 Addendum to the previous section

**Proposition 3.1** (The local expression of the Lie bracket). *Let  $(U; x^1, \dots, x^n)$  be a coordinate neighborhood of an  $n$ -manifold  $M$ . Then the Lie bracket of two vector fields*

$$X = \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}, \quad Y = \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j}$$

is expressed as

$$[X, Y] = \sum_{j=1}^n \left( \xi^k \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \xi^j}{\partial x^k} \right) \frac{\partial}{\partial x^j}.$$

*Proof.* For a smooth function  $f$  on  $U$ , it holds that

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f.$$

Hence  $[\partial/\partial x^i, \partial/\partial x^j] = 0$ . Then the conclusion follows from bilinearity of  $[X, Y]$  and the formula

$$[fX, Y] = f[X, Y] - (Yf)X, \quad [X, fY] = f[X, Y] + (Xf)Y$$

for a smooth function  $f$  and vector fields  $X$  and  $Y$ . □

**Proposition 3.2** (A local expression of the connection forms). *Let  $U$  be a domain of a Riemannian  $n$ -manifold  $(M, g)$  and  $[e_1, \dots, e_n]$  an orthonormal frame on  $U$ . Then the connection form  $\omega_i^j$  with respect to the frame  $[e_j]$  is obtained as*

$$\omega_i^j(e_k) = \frac{1}{2} \left( -\langle [e_i, e_j], e_k \rangle + \langle [e_j, e_k], e_i \rangle + \langle [e_k, e_i], e_j \rangle \right),$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product induced from  $g$ .

*Proof.* By the definition of the Levi-Civita connection  $\nabla$ ,

$$\begin{aligned} \omega_i^j(e_k) &= \langle \nabla_{e_k} e_i, e_j \rangle = e_k \langle e_i, e_j \rangle - \langle e_i, \nabla_{e_k} e_j \rangle = -\langle e_i, \nabla_{e_j} e_k + [e_k, e_j] \rangle \\ &= -e_j \langle e_i, e_k \rangle + \langle \nabla_{e_j} e_i, e_k \rangle - \langle e_i, [e_j, e_k] \rangle \\ &= \langle \nabla_{e_i} e_j, e_k \rangle + \langle [e_i, e_j], e_k \rangle - \langle e_i, [e_j, e_k] \rangle \\ &= e_i \langle e_j, e_k \rangle - \langle e_j, \nabla_{e_i} e_k \rangle + \langle [e_i, e_j], e_k \rangle - \langle e_i, [e_j, e_k] \rangle \\ &= -\langle e_j, \nabla_{e_k} e_i \rangle - \langle e_j, [e_i, e_k] \rangle + \langle [e_i, e_j], e_k \rangle - \langle e_i, [e_j, e_k] \rangle \\ &= -\omega_i^j(e_k) + \langle [e_i, e_j], e_k \rangle - \langle [e_j, e_k], e_i \rangle + \langle [e_k, e_i], e_j \rangle. \quad \square \end{aligned}$$

#### 3.2 Preliminaries

**Integrability condition, a review.** Let  $U$  be a domain of  $\mathbb{R}^m$  with coordinate system  $(x^1, \dots, x^m)$ , and consider a system of differential equations

$$(3.1) \quad \frac{\partial F}{\partial x^l} = F \Omega_l \quad (l = 1, \dots, m)$$

with initial condition

$$(3.2) \quad F(P_0) = F_0 \in M_n(\mathbb{R}), \quad P_0 = (x_0^1, \dots, x_0^m) \in U,$$

where  $F$  is an unknown map into the space of  $n \times n$ -real matrices  $M_n(\mathbb{R})$ , and the coefficient matrices  $\Omega_l$  ( $l = 1, \dots, m$ ) are  $M_n(\mathbb{R})$ -valued  $C^\infty$ -functions.

**Lemma 3.3.** *If the initial condition  $F_0$  in (3.2) is non-singular, i.e.,  $F_0 \in \text{GL}(n, \mathbb{R})^1$ ,  $F$  satisfying*

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<sup>1</sup> $\text{GL}(n, \mathbb{R})$  denotes the set of  $n \times n$ -regular matrices.

(3.1) is a  $\text{GL}(n, \mathbb{R})$ -valued function, that is,  $F$  is invertible for each point on  $U$ .

*Proof.* For each  $P \in U$ , take a smooth path  $\gamma(t) := (x^1(t), \dots, x^m(t))$  ( $0 \leq t \leq 1$ ) with  $\gamma(0) = P_0$  and  $\gamma(1) = P$ . Then the matrix-valued function  $\hat{F} := F \circ \gamma$  of one variable satisfies the ordinary differential equation

$$\frac{d\hat{F}}{dt} = \hat{F}\hat{\Omega}, \quad \hat{\Omega} := \sum_{l=1}^m \Omega_l \circ \gamma \frac{dx^l}{dt}.$$

Hence  $\varphi := \det \hat{F}$  satisfies

$$\frac{d\varphi}{dt} = \frac{d}{dt} \det \hat{F} = \text{tr} \left( \tilde{\hat{F}} \frac{d\hat{F}}{dt} \right) = \text{tr}(\tilde{\hat{F}}\hat{\Omega}) = \det \hat{F} \text{tr} \hat{\Omega} = \varphi \omega$$

where  $\tilde{\hat{F}}$  denotes the cofactor matrix of  $\hat{F}$  and  $\omega := \text{tr} \hat{\Omega}$ . So

$$\det \hat{F}(t) = \varphi(t) = \varphi_0 \exp \int_0^t \omega(\tau) d\tau \quad (\varphi_0 := \det F_0),$$

proving the lemma. □

As seen in the previous lectures<sup>2</sup>, the following *integrability condition* holds:

**Lemma 3.4.** *If a  $C^\infty$ -map  $F: U \rightarrow \text{GL}(n, \mathbb{R})$  satisfies (3.1), then it hold on  $U$  that*

$$(3.3) \quad \frac{\partial \Omega_l}{\partial x^k} - \frac{\partial \Omega_k}{\partial x^l} + \Omega_k \Omega_l - \Omega_l \Omega_k = O \quad (1 \leq k < l \leq m).$$

The integrability condition (3.3) guarantees existence of the solution of (3.1) as follows<sup>3</sup>:

**Theorem 3.5.** *Let  $\Omega_l: U \rightarrow \text{M}_m(\mathbb{R})$  ( $l = 1, \dots, m$ ) be  $C^\infty$ -functions defined on a simply connected domain  $U \subset \mathbb{R}^m$  satisfying (3.3) Then for each  $P_0 \in U$  and  $F_0 \in \text{M}_m(\mathbb{R})$ , there exists the unique  $m \times m$ -matrix valued function  $F: U \rightarrow \text{M}_m(\mathbb{R})$  satisfying (3.1) and (3.2). Moreover,*

- if  $F_0 \in \text{GL}(m, \mathbb{R})$ ,  $F(P) \in \text{GL}(m, \mathbb{R})$  holds on  $U$ ,
- if  $F_0 \in \text{SO}(m)$  and  $\Omega_l$ 's are skew-symmetric matrices,  $F(P) \in \text{SO}(m)$  holds on  $U$ .

**Coordinate-free expressions** Let  $\Omega_l: U \rightarrow \text{M}_n(\mathbb{R})$  ( $l = 1, \dots, m$ ) be  $C^\infty$ -functions defined on a domain  $U \subset \mathbb{R}^m$ , and define  $n \times n$ -matrix  $\Omega$  of 1-forms as

$$(3.4) \quad \Omega = \begin{pmatrix} \omega_1^1 & \omega_1^2 & \dots & \omega_1^n \\ \omega_2^1 & \omega_2^2 & \dots & \omega_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \omega_m^1 & \omega_m^2 & \dots & \omega_m^n \end{pmatrix} := \sum_{l=1}^m \Omega_l dx^l = \begin{pmatrix} \sum \omega_{l,1}^1 dx^l & \sum \omega_{l,2}^1 dx^l & \dots & \sum \omega_{l,n}^1 dx^l \\ \sum \omega_{l,1}^2 dx^l & \sum \omega_{l,2}^2 dx^l & \dots & \sum \omega_{l,n}^2 dx^l \\ \vdots & \vdots & \ddots & \vdots \\ \sum \omega_{l,1}^n dx^l & \sum \omega_{l,2}^n dx^l & \dots & \sum \omega_{l,n}^n dx^l \end{pmatrix},$$

where  $\Omega_l = (\omega_{i,j}^l)$ . Then  $\Omega$  is considered as a  $\text{M}_n(\mathbb{R})$ -valued 1-form, and (3.1) is restated as

$$(3.5) \quad dF = F\Omega.$$

**Lemma 3.6.** *Under the situation above, the integrability condition (3.3) is equivalent to*

$$(3.6) \quad d\Omega + \Omega \wedge \Omega = O, \quad \text{where} \quad \Omega \wedge \Omega = \left( \sum_{k=1}^n \omega_k^i \wedge \omega_j^k \right)_{i,j=1,\dots,n}.$$

<sup>2</sup>Proposition 2.3 in Advanced Topics in Geometry E (MTH.B501).

<sup>3</sup>Theorem 2.5 in Advanced Topics in Geometry E (MTH.B501).

*Proof.* Assume  $F$  be a solution of (3.5) with  $F \in \text{GL}(n, \mathbb{R})$ . Then

$$O = ddF = d(F\Omega) = dF \wedge \Omega + F d\Omega = F(\Omega \wedge \Omega + d\Omega). \quad \square$$

Thus, by using differential forms, we can state the system of partial differential equations (3.1) and its integrability condition (3.3) in coordinate-free form. The proof of Theorem 3.5 works not only simply connected domain  $U \subset \mathbb{R}^m$  but also simply connected  $m$ -manifold, and thus, we have

**Theorem 3.7.** *Let  $\Omega$  be an  $M_n(\mathbb{R})$ -valued 1-form on a simply connected  $m$ -manifold  $M$  satisfying (3.6). Then for each  $P_0 \in M$  and  $F_0 \in M_n(\mathbb{R})$ , there exists the unique  $n \times n$ -matrix valued function  $F: M \rightarrow M_n(\mathbb{R})$  satisfying (3.5) with  $F(P) = F_0$ . Moreover,*

- if  $F_0 \in \text{GL}(n, \mathbb{R})$ ,  $F(P) \in \text{GL}(n, \mathbb{R})$  holds on  $M$ ,
- if  $F_0 \in \text{SO}(m)$  and  $\Omega$  is skew-symmetric,  $F(P) \in \text{SO}(m)$  holds on  $M$ .

When  $n = 1$ , that is,  $\Omega$  is a usual 1-form,  $\Omega \wedge \Omega$  always vanishes, and the integrability condition (3.6) is simply  $d\Omega = 0$ . Then we have the following Poncaré's lemma<sup>4</sup>.

**Theorem 3.8** (Poincaré's lemma). *If a differential 1-form  $\omega$  defined on a simply connected and connected  $m$ -manifold  $M$  is closed, that is,  $d\omega = 0$  holds, then there exists a  $C^\infty$ -function  $f$  on  $U$  such that  $df = \omega$ . Such a function  $f$  is unique up to additive constants.*

*Proof.* Since  $\omega$  is closed, there exists a function  $F$  on  $M$  satisfying  $dF = F\omega$  with initial condition  $F(P_0) = 1$ . By Lemma 3.3,  $F$  does not vanish on  $M$ , that is,  $F > 0$ . Hence  $f := \log F$  is a smooth function on  $M$  satisfying  $df = dF/F = F\omega/F = \omega$ . Take another function  $g$  on  $M$  satisfying  $dg = \omega$ ,  $d(f - g) = 0$  holds. Then connectedness of  $M$  infers that  $f - g$  is constant.  $\square$

### 3.3 Curvature form

Let  $U$  be a domain of  $n$ -dimensional Riemannian manifold  $(M, g)$ . We let  $\Omega$  be the connection form with respect to an orthonormal frame  $[e_1, \dots, e_n]$  on  $U$ , as defined in Definition 2.15.

**Definition 3.9.** We define a skew-symmetric matrix-valued 2-form by  $K := d\Omega + \Omega \wedge \Omega$  and call the *curvature form* with respect to the frame  $[e_1, \dots, e_n]$ .

Take an orthonormal frame  $[v_1, \dots, v_n]$  on  $U$  and take a gauge transformation  $\Theta: U \rightarrow O(n)$ :

$$[e_1, \dots, e_n] = [v_1, \dots, v_n]\Theta.$$

Denoting the connection form and the curvature form with respect to  $[v_j]$  by  $\tilde{\Omega}$  and  $\tilde{K}$ . Then

**Proposition 3.10.** (1)  $\Omega = \Theta^{-1}\tilde{\Omega}\Theta + \Theta^{-1}d\Theta$ , (2)  $K = \Theta^{-1}\tilde{K}\Theta$ .

*Proof.* Since

$$\begin{aligned} [e_1, \dots, e_n]\Omega &= \nabla[e_1, \dots, e_n] = \nabla([v_1, \dots, v_n]\Theta) = \nabla[v_1, \dots, v_n]\Theta + [v_1, \dots, v_n]d\Theta \\ &= [v_1, \dots, v_n]\tilde{\Omega}\Theta + [v_1, \dots, v_n]d\Theta = [e_1, \dots, e_n]\Theta^{-1}(\tilde{\Omega}\Theta + d\Theta), \end{aligned}$$

the first assertion is obtained. Next, noticing  $d(\tilde{\Omega}\Theta) = (d\tilde{\Omega})\Theta - \tilde{\Omega} \wedge d\Theta$ ,  $\tilde{\Omega}\Theta^{-1} \wedge \Theta\tilde{\Omega} = \tilde{\Omega} \wedge \tilde{\Omega}$ , and so on, we have

$$\begin{aligned} d\Omega + \Omega \wedge \Omega &= d(\Theta^{-1}\tilde{\Omega}\Theta + \Theta^{-1}d\Theta) + (\Theta^{-1}\tilde{\Omega}\Theta + \Theta^{-1}d\Theta) \wedge (\Theta^{-1}\tilde{\Omega}\Theta + \Theta^{-1}d\Theta) \\ &= -\Theta^{-1}d\Theta\Theta^{-1}\tilde{\Omega}\Theta + \Theta^{-1}d\tilde{\Omega}\Theta - \Theta^{-1}\tilde{\Omega} \wedge d\Theta - \Theta^{-1}d\Theta\Theta^{-1} \wedge d\Theta \\ &\quad + \Theta^{-1}\tilde{\Omega}\Theta \wedge \Theta^{-1}\tilde{\Omega}\Theta + \Theta^{-1}d\Theta \wedge \Theta^{-1}\tilde{\Omega}\Theta + \Theta^{-1}\tilde{\Omega}\Theta \wedge \Theta^{-1}d\Theta + \Theta^{-1}d\Theta \wedge \Theta^{-1}d\Theta \\ &= \Theta^{-1}(d\tilde{\Omega} + \tilde{\Omega} \wedge \tilde{\Omega})\Theta, \end{aligned}$$

proving (2).  $\square$

<sup>4</sup>Theorem 2.6 in Advanced Topics in Geometry E (MTH.B501).

The goal of this section is to prove the following

**Theorem 3.11.** *Let  $U$  be a domain of a Riemannian  $n$ -manifold  $(M, g)$  and  $K$  the curvature form with respect to an orthonormal frame  $[e_1, \dots, e_n]$  on  $U$ . For a point  $P \in U$ , there exists a local coordinate system  $(x^1, \dots, x^n)$  around  $P$  such that  $[\partial/\partial x^1, \dots, \partial/\partial x^n]$  is an orthonormal frame if and only if  $K$  vanishes on a neighborhood of  $P$ .*

*Remark 3.12.* By (2) of Proposition 3.10, the condition  $K = 0$  does not depend on choice of orthonormal frames. A Riemannian manifold  $(M, g)$  said to be *flat* if  $K = 0$  holds on  $M$ .

*Proof of Theorem 3.11.* First, we shall show the “only if” part: Let  $(x^1, \dots, x^n)$  be a coordinate system such that  $[e_j := \partial/\partial x^j]$  is an orthonormal frame. Since

$$[e_j, e_k] = \left[ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right] = \mathbf{0},$$

Proposition 3.2 yields that all components of the connection forms  $\omega_i^j$  vanish. Hence we have  $K = 0$ .

Conversely, assume  $K = 0$  for an orthonormal frame  $[e_j]$ . Since the connection form  $\Omega$  satisfies  $d\Omega + \Omega \wedge \Omega = 0$ , there exists a matrix-valued function  $\Theta: V \rightarrow \text{SO}(n)$  satisfying  $d\Theta = \Omega\Theta$ ,  $\Theta(P) = \text{id}$  on a sufficiently small neighborhood  $V$  of  $P$ , because of Theorem 3.5. Take a new orthonormal frame  $[v_1, \dots, v_n] := [e_1, \dots, e_n]\Theta$ . Then by (1) of Proposition 3.10, the connection form  $\tilde{\Omega} = (\tilde{\omega}_i^j)$  with respect to  $[v_j]$  vanishes identically. So by Lemma 2.17,  $d\omega^i = 0$  holds for  $i = 1, \dots, n$ . Hence by the Poincaré Lemma (Theorem 3.8), there exists a smooth functions on a neighborhood  $V$  of  $P$ . Such  $(x^1, \dots, x^n)$  is a desired coordinate system if  $V$  is sufficiently small.  $\square$

### Exercises

**3-1** Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2 d\theta^2 \quad \text{on} \quad U := \{(r, \theta); 0 < r < r_0, -\pi < \theta < \pi\},$$

where  $r_0 \in (0, +\infty]$  and  $\varphi$  is a positive smooth function defined on  $(0, r_0)$  with

$$\lim_{r \rightarrow +0} \varphi(r) = 0, \quad \lim_{r \rightarrow +0} \varphi'(r) = 1.$$

Find a function  $\varphi$  such that  $(U, g)$  is flat. (Hint:  $[\partial/\partial r, (1/\varphi)\partial/\partial\theta]$  is an orthonormal frame.)

**3-2** Compute the curvature form of  $H^2(-c^2)$  with respect to an orthonormal frame  $[e_1, e_2]$  as in Exercise 2-2.