4 The Sectional Curvature

4.1 Preliminaries

Exterior derivatives. Let α and ω be a 2-form and 1-form on a manifold M, respectively. The exterior product of α and ω is defined as a 3-form on M by

$$(\alpha\wedge\omega)(X,Y,Z)=(\omega\wedge\alpha)(X,Y,Z):=\alpha(X,Y)\omega(Z)+\alpha(Y,Z)\omega(X)+\alpha(Z,X)\omega(Y). \tag{4.1}$$

Then by a direct computation together with (2.7), it holds that

$$(\mu\wedge\omega)\wedge\lambda=\mu\wedge(\omega\wedge\lambda)\Bigl(=:\mu\wedge\omega\wedge\lambda\Bigr) \tag{4.2}$$

for 1-forms μ, ω and λ. The *exterior derivative* of a 2-from α is a 3-form $d\alpha$ defined as

$$\begin{aligned} &d\alpha(X,Y,Z)\\ &\quad:=X\alpha(Y,Z)d+Y\alpha(Z,X)+Z\alpha(X,Y)-\alpha([X,Y],Z)-\alpha([Z,X],Y)-\alpha([Y,Z],X). \end{aligned} \tag{4.3}$$

Then, for one forms μ and ω, we have

$$dd\omega=0,\qquad d(\mu\wedge\omega)=d\mu\wedge\omega-\mu\wedge d\omega, \tag{4.4}$$

by the definition and the Jacobi identity (2.3).

Exterior products of tangent vectors. Let V be an n-dimensional vector space ($1\leqq n<\infty$) and denote by V^* its dual. Then $(V^*)^*$ can be naturally identified with V itself. In fact,

$$I: V\ni \boldsymbol{v}\longmapsto I_{\boldsymbol{v}}\in(V^*)^*:=\{A\colon V^*\to\mathbb{R};\text{linear}\},\qquad I_{\boldsymbol{v}}(\alpha):=\alpha(\boldsymbol{v})$$

is a linear map with trivial kernel. Then I is an isomorphism because $\dim(V^*)^*=\dim V$.

We denote by $\wedge^2V:=\wedge^2(V^*)^*$ the set of skew-symmetric bilinear forms on V^*. For vectors $\boldsymbol{v}$, $\boldsymbol{w}\in V$, the *exterior product* of them is an element of $\wedge^2V$ defined as

$$(\boldsymbol{v}\wedge\boldsymbol{w})(\alpha,\beta):=\alpha(\boldsymbol{v})\beta(\boldsymbol{w})-\alpha(\boldsymbol{w})\beta(\boldsymbol{v})\qquad(\alpha,\beta\in V^*).$$

For a basis $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ on V,

$$\{\boldsymbol{e}_i\wedge\boldsymbol{e}_j;1\leqq i<j\leqq n\} \tag{4.5}$$

is a basis of $\wedge^2V$. In particular $\dim\wedge^2V=\frac{1}{2}n(n-1)$. When V is a vector space endowed with an inner product $\langle\ ,\ \rangle$ and $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ is an orthonormal basis, there exists the unique inner product, which is also denoted by $\langle\ ,\ \rangle$, of $\wedge^2V$ such that (4.5) is an orthonormal basis. This definition of the inner product does not depend on choice of orthonormal bases of V. In fact, take another orthonormal basis $[\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]$ related with $[\boldsymbol{e}_j]$ by

$$[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]=[\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]\Theta\qquad\Theta=(\theta^j_i)\in\mathrm{O}(n).$$

Since ${}^t\Theta=\Theta^{-1}$, $[\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]=[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]{}^t\Theta$ holds. Hence

$$\boldsymbol{v}_s\wedge\boldsymbol{v}_t=\left(\sum_i\theta^i_s\boldsymbol{e}_i\right)\wedge\left(\sum_j\theta^j_t\boldsymbol{e}_j\right)=\sum_{i,j}\theta^s_i\theta^t_j(\boldsymbol{e}_i\wedge\boldsymbol{e}_j)=\sum_{i<j}\left(\theta^s_i\theta^t_j-\theta^s_j\theta^t_i\right)(\boldsymbol{e}_i\wedge\boldsymbol{e}_j),$$

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and so

$$\begin{aligned}
\langle \boldsymbol{v}_s\wedge\boldsymbol{v}_t,\boldsymbol{v}_u\wedge\boldsymbol{v}_v\rangle &= \sum_{i<j,k<l}(\theta_i^s\theta_j^t-\theta_j^s\theta_i^t)(\theta_k^u\theta_l^v-\theta_l^u\theta_k^v)\,\langle \boldsymbol{e}_i\wedge\boldsymbol{e}_j,\boldsymbol{e}_k\wedge\boldsymbol{e}_l\rangle\\
&= \sum_{i<j,k<l}(\theta_i^s\theta_j^t-\theta_j^s\theta_i^t)(\theta_k^u\theta_l^v-\theta_l^u\theta_k^v)\delta_{ik}\delta_{jl} = \sum_{i<j}(\theta_i^s\theta_j^t-\theta_j^s\theta_i^t)(\theta_i^u\theta_j^v-\theta_j^u\theta_i^v)\\
&= \sum_{i<j}(\theta_i^s\theta_j^t\theta_i^u\theta_j^v-\theta_j^s\theta_i^t\theta_i^u\theta_j^v-\theta_i^s\theta_j^t\theta_j^u\theta_i^v+\theta_j^s\theta_i^t\theta_j^u\theta_i^v)\\
&= \sum_{i<j}\theta_i^s\theta_j^t\theta_i^u\theta_j^v+\sum_{i<j}\theta_j^s\theta_i^t\theta_i^u\theta_j^v-\sum_{i>j}\theta_j^s\theta_i^t\theta_i^u\theta_j^v+\sum_{i>j}\theta_i^s\theta_j^t\theta_i^u\theta_j^v\\
&= \sum_{i\neq j}\theta_i^s\theta_j^t\theta_i^u\theta_j^v-\sum_{i\neq j}\theta_j^s\theta_i^t\theta_i^u\theta_j^v\\
&= \sum_{i,j}(\theta_i^s\theta_j^t\theta_i^u\theta_j^v-\theta_j^s\theta_i^t\theta_i^u\theta_j^v)-\sum_i(\theta_i^s\theta_i^t\theta_i^u\theta_i^v-\theta_i^s\theta_i^t\theta_i^u\theta_i^v)\\
&= \delta^{su}\delta^{tv}-\delta^{tu}\delta^{sv}
\end{aligned}$$

because $\sum_i\theta_i^s\theta_i^t=\delta^{st}$. So, if $s<t$ and $u<v$, the second term of the right-hand side vanishes. That is, $\{\boldsymbol{v}_s\wedge\boldsymbol{v}_t\,;\,s<t\}$ is an orthonormal basis as well as $\{\boldsymbol{e}_i\wedge\boldsymbol{e}_j\,;\,i<j\}$ is.

Symmetric bilinear forms. Let V be a real vector space. A bilinear map $q\colon V\times V\to\mathbb{R}$ is said to be *symmetric* if $q(\boldsymbol{v},\boldsymbol{w})=q(\boldsymbol{w},\boldsymbol{v})$ for all $\boldsymbol{v}$, $\boldsymbol{w}\in V$.

Lemma 4.1. *Two symmetric bilinear forms q and q' coincide with each other if and only if $q(\boldsymbol{v},\boldsymbol{v})=q'(\boldsymbol{v},\boldsymbol{v})$ hold for all $\boldsymbol{v}\in V$.*

Proof. By symmetricity, $q(\boldsymbol{v},\boldsymbol{w})=\frac{1}{2}(q(\boldsymbol{v}+\boldsymbol{w},\boldsymbol{v}+\boldsymbol{w})-q(\boldsymbol{v},\boldsymbol{v})-q(\boldsymbol{w},\boldsymbol{w}))$ holds. □

4.2 Sectional Curvature

Let U be a domain on a Riemannian n-manifold (M,g), and $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ an orthonormal frame on U. Denote by $(\omega^j)_{j=1,\dots,n}$, $\varOmega=(\omega_i^j)_{i,j=1,\dots,n}$ and $K=(\kappa_i^j)_{i=1,\dots,n}:=d\varOmega+\varOmega\wedge\varOmega$ the dual frame, the connection form and the curvature form with respect to the frame $[\boldsymbol{e}_j]$. Then Lemma 2.17 and Definition 3.9, we have

$$d\omega^j=\sum_l\omega^l\wedge\omega_l^j,\qquad \kappa_i^j=d\omega_i^j+\sum_l\omega_l^j\wedge\omega_i^l. \tag{4.6}$$

Since $\varOmega$ is a one form valued in the skew-symmetric matrices, so is K:

$$\omega_i^j=-\omega_j^i,\qquad \kappa_i^j=-\kappa_j^i. \tag{4.7}$$

Proposition 4.2 (The first Bianchi identity). $\kappa_j^i(\boldsymbol{e}_k,\boldsymbol{e}_l)+\kappa_k^i(\boldsymbol{e}_l,\boldsymbol{e}_j)+\kappa_l^i(\boldsymbol{e}_j,\boldsymbol{e}_k)=0.$

Proof. By (4.6) and (4.4),

$$\begin{aligned}
0=dd\omega^i=d\left(\sum_s\omega^s\wedge\omega_s^i\right)&=\sum_s\left(d\omega^s\wedge\omega_s^i-\omega^s\wedge\omega_s^i\right)\\
&=\sum_s\left(\sum_m(\omega^m\wedge\omega_m^s)\wedge\omega_s^i-\omega^s\wedge\left(\kappa_s^i-\sum_m\omega_m^i\wedge d\omega_s^m\right)\right)\\
&=\sum_{s,m}\omega^m\wedge\omega_m^s\wedge\omega_s^i+\sum_{s,m}\omega^s\wedge\omega_m^i\wedge\omega_s^m-\sum_s\omega^s\wedge\kappa_s^i\\
&=\sum_{s,m}\omega^m\wedge(\omega_m^s\wedge\omega_s^i+\omega_s^i\wedge\omega_m^s)-\sum_s\omega^s\wedge\kappa_s^i=-\sum_s\omega^s\wedge\kappa_s^i.
\end{aligned}$$

Hence

$$\begin{aligned} 0 &= \sum_s (\omega^s \wedge \kappa^i_s)(\boldsymbol{e}_j, \boldsymbol{e}_k, \boldsymbol{e}_l) = \sum_s \bigl(\omega^s(\boldsymbol{e}_j)\kappa^i_s(\boldsymbol{e}_k, \boldsymbol{e}_l) + \omega^s(\boldsymbol{e}_k)\kappa^i_s(\boldsymbol{e}_l, \boldsymbol{e}_j) + \omega^s(\boldsymbol{e}_l)\kappa^i_s(\boldsymbol{e}_j, \boldsymbol{e}_k)\bigr) \\ &= \sum_s \bigl(\delta^s_j \kappa^i_s(\boldsymbol{e}_k, \boldsymbol{e}_l) + \delta^s_k \kappa^i_s(\boldsymbol{e}_l, \boldsymbol{e}_j) + \delta^s_l \kappa^i_s(\boldsymbol{e}_j, \boldsymbol{e}_k)\bigr) \\ &= \kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) + \kappa^i_k(\boldsymbol{e}_l, \boldsymbol{e}_j) + \kappa^i_l(\boldsymbol{e}_j, \boldsymbol{e}_k), \end{aligned}$$

proving the assertion. □

Corollary 4.3. $\kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) = \kappa^k_l(\boldsymbol{e}_i, \boldsymbol{e}_j)$.

Proof. By Proposition 4.2,

$$\begin{aligned} \kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) + \kappa^i_k(\boldsymbol{e}_l, \boldsymbol{e}_j) + \kappa^i_l(\boldsymbol{e}_j, \boldsymbol{e}_k) = 0 \\ \kappa^j_k(\boldsymbol{e}_i, \boldsymbol{e}_l) + \kappa^j_i(\boldsymbol{e}_l, \boldsymbol{e}_k) + \kappa^j_l(\boldsymbol{e}_k, \boldsymbol{e}_i) = 0 \\ \kappa^k_i(\boldsymbol{e}_j, \boldsymbol{e}_l) + \kappa^k_j(\boldsymbol{e}_l, \boldsymbol{e}_i) + \kappa^k_l(\boldsymbol{e}_i, \boldsymbol{e}_j) = 0. \end{aligned}$$

Summing up these and noticing $\kappa^j_i = -\kappa^i_j$, we have the conclusion. □

A quadratic form induced from the curvature form. We fix a point $p \in U$. Under the notation above, we can define a bilinear map

$$\text{(4.8)} \qquad \boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j, k<l} \kappa^j_i(\boldsymbol{e}_k, \boldsymbol{e}_l)\xi^{kl}\eta^{ij}, \qquad \boldsymbol{\xi} = \sum_{k<l} \xi^{kl}\boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j} \eta^{ij}\boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

on $\wedge^2 T_pM$, where $\boldsymbol{e}_j$, κ^j_i...are considered tangent vectors, 2-forms at the fixed point p. In fact, one can show that the definition (4.8) is independent of choice of orthonormal frames. As a immediate conclusion of Corollary 4.3, we have

Lemma 4.4. $\boldsymbol{K}$ *is symmetric.*

Hence, taking Lemma 4.1 into an account, we define the sectional curvature as follows:

Definition 4.5. Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in T_pM. The *sectional curvature* of (M, g) with respect to the plane Π_p is a number

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of Π_p

Remark 4.6. For (not necessarily orthonormal) basis $\{\boldsymbol{x}, \boldsymbol{y}\}$ of Π_p, the sectional curvature is expressed as

$$K(\Pi_p) = \frac{\boldsymbol{K}(\boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{x} \wedge \boldsymbol{y})}{\langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{x} \wedge \boldsymbol{y} \rangle},$$

where $\langle\ ,\ \rangle$ of the right-hand side is the inner product of $\wedge^2 T_pM$ induced from the Riemannian metric.

Remark 4.7. The sectional curvature is a scalar corresponding to a 2-plane in the tangent space T_pM. Hence it can be considered as a function of 2-Grassmanian bundle induced from the tangent bundle TM.

Exercises

4-1 Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \qquad \text{on} \qquad U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ and φ is a positive smooth function defined on $(0, r_0)$ with

$$\lim_{r\to+0} \varphi(r) = 0, \qquad \lim_{r\to+0} \frac{\varphi(r)}{r} = 1.$$

Classify the function φ so that g is of constant sectional curvature.

4-2 Let $M \subset \mathbb{R}^{n+1}$ be an embedded submanifold endowed with the Riemannian metric induced from the canonical Euclidean metric of $\mathbb{R}^{n+1}$. Then the position vector $\boldsymbol{x}(p)$ of $p \in M$ induces a smooth map

$$\boldsymbol{x} \colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$-tuple of C^∞-functions. Let $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ be an orthonormal frame defined on a domain $U \subset M$. Since $T_pM \subset \mathbb{R}^{n+1}$, we can consider that $\boldsymbol{e}_j$ is a smooth map from $U \to \mathbb{R}^{n+1}$. Take a dual basis (ω^j) to $[\boldsymbol{e}_j]$. Prove that

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j$$

holds on U. Here, we regard that $d\boldsymbol{x}$ is an $(n+1)$-tuple of differential forms and $\boldsymbol{e}_j$ is an $\mathbb{R}^{n+1}$-valued function for each j.