5 Space forms

5.1 Constant sectional curvature

Let (M, g) be a Riemannian n-manifold, and let

$$\mathrm{Gr}_2(TM) := \cup_p \mathrm{Gr}_2(T_pM),$$
$$\mathrm{Gr}_2(T_pM) := \text{2-Grassmannian of } T_pM = \{\Pi_p \subset T_pM\,;\ \text{2-dimensional subspace}\}.$$

The sectional curvature defined in Definition 4.5 is a map $K\colon \mathrm{Gr}_2(TM) \to \mathbb{R}$ such that

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is the orthonormal basis of Π_p.

Fix a point p, and take an orthornormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ defined on a neighborhood U of p. Denote by (ω^j), $\Omega = (\omega_i^j)$ and $K = (\kappa_i^j)$ the dual frame, the connection form and the curvature form with respect to the frame $[\boldsymbol{e}_j]$, respectively.

Theorem 5.1. *Assume there exists a real number k such that $K(\Pi_p) = k$ for all 2-dimensional subspace $\Pi_p \in T_pM$ for a fixed p. Then the curvature form is expressed as*

$$\kappa_j^i = k\omega^i \wedge \omega^j.$$

Conversely, the curvature form is written as above, the sectional curvature at p is constant k.

Proof. By the assumption, $k = K(\mathrm{Span}\{\boldsymbol{e}_i, \boldsymbol{e}_j\}) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_j, \boldsymbol{e}_i \wedge \boldsymbol{e}_j)] = \kappa_j^i(\boldsymbol{e}_i, \boldsymbol{e}_j)$. Let

$$\boldsymbol{v} := \cos\theta \boldsymbol{e}_i + \sin\theta \boldsymbol{e}_j, \qquad \boldsymbol{w} := \cos\varphi \boldsymbol{e}_l + \sin\varphi \boldsymbol{e}_m$$

where $\{i, j\} \neq \{l, m\}$, and set $\Pi_{\theta,\varphi} := \mathrm{Span}\{\boldsymbol{v}, \boldsymbol{w}\} \subset T_pM$. Then by biliniearity of the $\wedge$-product on T_pM, it holds that

$$\boldsymbol{v} \wedge \boldsymbol{w} = \cos\theta\cos\varphi \boldsymbol{e}_i \wedge \boldsymbol{e}_l + \cos\theta\sin\varphi \boldsymbol{e}_i \wedge \boldsymbol{e}_m + \sin\theta\cos\varphi \boldsymbol{e}_j \wedge \boldsymbol{e}_l + \sin\theta\sin\varphi \boldsymbol{e}_j \wedge \boldsymbol{e}_m.$$

Since $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of $\Pi_{\theta,\varphi}$, bilinieartity and symmetricity of $\boldsymbol{K}$ implies

$$\begin{aligned}
k =& K(\Pi_{\theta,\varphi}) = \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}) \\
=& \cos^2\theta\cos^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_i \wedge \boldsymbol{e}_l) + \cos^2\theta\sin^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_i \wedge \boldsymbol{e}_m) \\
&+ \sin^2\theta\cos^2\varphi \boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) + \sin^2\theta\sin^2\varphi \boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) \\
&+ 2\cos^2\theta\cos\varphi\sin\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_i \wedge \boldsymbol{e}_m) + 2\cos\theta\sin\theta\cos^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) \\
&+ 2\cos\theta\sin\theta\cos\varphi\sin\varphi(\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l)) \\
&+ 2\cos\theta\sin\theta\sin^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + 2\sin^2\theta\cos\varphi\sin\varphi \boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) \\
=& k + 2\big(\cos^2\theta\cos\varphi\sin\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_i \wedge \boldsymbol{e}_m) + \cos\theta\sin\theta\cos^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) \\
&+ \cos\theta\sin\theta\cos\varphi\sin\varphi(\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l)) \\
&+ \cos\theta\sin\theta\sin^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \sin^2\theta\cos\varphi\sin\varphi \boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m)\big).
\end{aligned} \tag{5.1}$$

So, by letting $\theta = 0$, we have

$$\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = 0. \tag{5.2}$$

Similarly, letting $\theta = \pi/2$, $\varphi = 0$ and $\varphi = \pi/2$, we have $\boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = 0$. Hence the equality (5.1) implies

$$\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = 0.$$

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By definition (4.8), this is equivalent to

$$\kappa_j^m(\boldsymbol{e}_i,\boldsymbol{e}_l)+\kappa_j^l(\boldsymbol{e}_i,\boldsymbol{e}_m)=-(\kappa_m^j(\boldsymbol{e}_i,\boldsymbol{e}_l)+\kappa_l^j(\boldsymbol{e}_i,\boldsymbol{e}_m)).$$

Then by Proposition 4.2, we have

$$0=\kappa_m^j(\boldsymbol{e}_i,\boldsymbol{e}_l)+\kappa_l^j(\boldsymbol{e}_i,\boldsymbol{e}_m)=\kappa_m^j(\boldsymbol{e}_i,\boldsymbol{e}_l)-\kappa_i^j(\boldsymbol{e}_m,\boldsymbol{e}_l)-\kappa_m^j(\boldsymbol{e}_l,\boldsymbol{e}_i)=2\kappa_m^j(\boldsymbol{e}_i,\boldsymbol{e}_l)-\kappa_i^j(\boldsymbol{e}_m,\boldsymbol{e}_l).$$

Exchanging the roles of i and m, it holds that $2\kappa_i^j(\boldsymbol{e}_m,\boldsymbol{e}_l)-\kappa_m^j(\boldsymbol{e}_i,\boldsymbol{e}_l)=0$. So we have

$$\kappa_i^j(\boldsymbol{e}_m,\boldsymbol{e}_l)=0 \qquad (\text{if } \{i,j\}\neq\{m,l\}). \tag{5.3}$$

On the other hand, (5.2) means that $\kappa_i^j(\boldsymbol{e}_i,\boldsymbol{e}_l)=\kappa_i^j(\boldsymbol{e}_j,\boldsymbol{e}_l)=0$ when $l\neq i,j$. Summing up, we have

$$\kappa_i^j(\boldsymbol{e}_k,\boldsymbol{e}_l)=\begin{cases}k & (i,j)=(k,l)\\ 0 & \text{otherwise},\end{cases}$$

proving the theorem. □

We now consider the case that the assumption of Theorem 5.1 holds for each $p\in M$.

Theorem 5.2. *Assume that for each p, there exists a real number $k(p)$ such that $K(\Pi_p)=k(p)$ for any $\Pi_p\in \mathrm{Gr}_2(T_pM)$. Then the function $k\colon M\ni p\to k(p)\in\mathbb{R}$ is constant provided that M is connected.*

Proof. Take the exterior derivative of the definition $\kappa_i^j=d\omega_i^j+\sum_s\omega_s^j\wedge\omega_i^s$, it holds that

$$\begin{aligned}d\kappa_i^j&=d(d\omega_i^j)+\sum_s d\omega_s^j\wedge\omega_i^s-\sum_s s\omega_s^j\wedge d\omega_i^s\\&=\sum_s\left(\kappa_s^j-\sum_t\omega_t^j\wedge\omega_s^t\right)\wedge\omega_i^s-\sum_s\omega_s^j\wedge\left(\kappa_i^s-\sum_t\omega_t^s\wedge\omega_i^t\right),\end{aligned}$$

and hence we have the identity

$$d\kappa_i^j=\sum_s\left(\kappa_s^j\wedge\omega_i^s-\omega_s^j\wedge\kappa_i^s\right), \tag{5.4}$$

which is known as the *second Bianchi identity*. By our assumption, Theorem 5.1 implies that $\kappa_i^j=k\omega^i\wedge\omega^j$. Then by Lemma 2.17,

$$\begin{aligned}d\kappa_i^j&=d(k\omega^i)\wedge\omega^j-k\omega^i\wedge d\omega^j=dk\wedge\omega^i\wedge\omega^j+kd\omega^i\wedge\omega^j-k\omega^i\wedge d\omega^j\\&=dk\wedge\omega^i\wedge\omega^j+\sum_s k\omega^s\wedge\omega_s^i\wedge\omega^j-\sum_s k\omega^i\wedge\omega^s\wedge\omega_s^j=dk\wedge\omega^i\wedge\omega^j+d\kappa_i^j\end{aligned}$$

holds for each i and j. Thus, $dk\wedge\omega^i\wedge\omega^j=0$ for all i and j, which implies $dk=0$. This equality is independent of choice of orthonormal frames. Since M is connected, k is constant. □

5.2 Space forms

Let (M,g) be a Riemannian n-manifold. A path $\gamma\colon[0,+\infty)\to M$ is said to be a *divergence path* if for any compact subset $K\in M$, there exists $t_0\in(0,+\infty)$ such that $\gamma([t_0,+\infty))\subset M\setminus K$. If any divergent path has infinite length, (M,g) is said to be complete. [5]In particular, a compact Riemannian manifold (without boundary) is automatically complete.

[5]Usually, completeness is defined in terms of geodesics: A Riemannian manifold (M,g) is complete if any geodesics are defined on entire $\mathbb{R}$. The definition here is one of the equivalent conditions of completeness, expressed in the *Hopf-Rinow theorem*.

Definition 5.3. An n-dimensional *space form* is a complete Riemannian n-manifold of constant sectional curvature.

Example 5.4. The Euclidean n-space is a space form of constant sectional curvature 0. In fact, let $(x^1,\dots,x^n)$ be the canonical Cartesian coordinate system and set $\boldsymbol{e}_j = \partial/\partial x^j$. Then $[\boldsymbol{e}_j]$ is an orthornormal frame defined on the entire $\mathbb{R}^n$, and Propositions 3.1 and 3.2 implies that the connection form $\omega^i_j = 0$. Hence the curvature forms vanish, and then the sectional curvature is identically zero.

So it is sufficient to show completeness. Let $\gamma\colon [0,+\infty) \to \mathbb{R}^n$ be a divergent path. Then for each $r > 0$, there exists $t_0 > 0$ such that $|\gamma(t)| > r$ holds on $[t_0,+\infty)$, equivalently, $|\gamma(t)| \to +\infty$ as $t \to +\infty$. So the length L of the curve γ is

$$L = \lim_{t\to+\infty} \int_0^t |\dot\gamma(\tau)|\,d\tau \geqq \lim_{t\to+\infty} \left| \int_0^t \dot\gamma(\tau)\,d\tau \right| = \lim_{t\to+\infty} |\gamma(t)-\gamma(0)| \geqq \lim_{t\to+\infty} |\gamma(t)| - |\gamma(0)| = +\infty.$$

Here, we used the triangle inequality of integrals for vector-valued functions[6].

5.3 The Hyperbolic spaces

Let $H^n(-c^2)$ be the hyperbolic n-space defined in Example 1.8, where c is a non-zero constant:

$$H^n(-c^2) := \left\{ \boldsymbol{x} = (x^0,\dots,x^n) \in \mathbb{R}^{n+1}_1 \;\middle|\; \langle \boldsymbol{x},\boldsymbol{x}\rangle_L = -\frac{1}{c^2}, cx_0 > 0 \right\},$$

where $(\mathbb{R}^{n+1}_1, \langle\ ,\ \rangle_L)$ be the Lorentz-Minkowski $(n+1)$-space as in Example 1.7. The tangent space $T_{\boldsymbol{x}} H^n(-c^2)$ is the orthogonal complement $\boldsymbol{x}^\perp$ of $\boldsymbol{x}$, and the restriction g_H of the inner product $\langle\ ,\ \rangle_L$ to $T_{\boldsymbol{x}} H^n(-c^2)$ is positive definite. Thus, $(H^n(-c^2), g_H)$ is a Riemannian manifold, called the *hyperbolic n-space*.

Theorem 5.5. *The hyperbolic space $(H^n(-c^2), g_H)$ is of constant sectional curvature $-c^2$.*

Proof. Notice that $H^n(-c^2)$ can be expressed as a graph $x^0 = \frac{1}{c}\sqrt{1 + c^2\left((x^1)^2 + \dots + (x^n)^2\right)}$ defined on the $(x^1,\dots,x^n)$-hyperplane, it is covered by single chart. Then there exists a orthonormal frame field $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ defined on entire $H^n(-c^2)$. Denote by (ω^i), $\Omega = (\omega^j_i)$ and $K = (\kappa^j_i)$ the dual frame, the connection form and the curvature form with respect to $[\boldsymbol{e}_j]$, respectively.

Regarding $T_{\boldsymbol{x}} H^n(-c^2)$ as a linear subspace in $\mathbb{R}^{n+1}_1$, we can consider $\boldsymbol{e}_j$ as a vector-valued function. In addition the position vector $\boldsymbol{x} \in H^n(-c^2)$ can be also regarded as a vector-valued function. Since $T_{\boldsymbol{x}} H^n(-c^2) = \boldsymbol{x}^\perp$,

$$\mathcal{F} := (\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n)\colon H^n(-c^2) \to \mathrm{M}_{n+1}(\mathbb{R}) \qquad \boldsymbol{e}_0 = c\boldsymbol{x} \tag{5.5}$$

gives a pseudo orthornormal frame along $H^n(-c^2)$, that is, ${}^t\mathcal{F}Y\mathcal{F} = Y$ $(Y := \mathrm{diag}(-1,1,\dots,1))$ holds.

As seen in Exercise 4-2, it holds that

$$d\boldsymbol{e}_0 = c\,d\boldsymbol{x} = c\sum_{j=1}^n \omega^j \boldsymbol{e}_j. \tag{5.6}$$

On the other hand, for each $j = 1,\dots,n$, decompose the vector-valued one form $d\boldsymbol{e}_j$ as

$$d\boldsymbol{e}_j = h_j \boldsymbol{e}_0 + \sum_s \alpha^s_j \boldsymbol{e}_s,$$

[6]See, for example, Theorem A.1.4 in [UY17] for $n = 2$. The idea of the proof works for general n.

where h_j and α_j^s are one forms on $H^n(-c^2)$. Here,

$$h_j = -\langle d\boldsymbol{e}_j, \boldsymbol{e}_0\rangle_L = -d\langle \boldsymbol{e}_j, \boldsymbol{e}_0\rangle_L + \langle \boldsymbol{e}_j, d\boldsymbol{e}_0\rangle_L = c\omega^j,$$

and

$$\alpha_j^s = \langle d\boldsymbol{e}_j, \boldsymbol{e}_s\rangle_L = d\langle \boldsymbol{e}_j, \boldsymbol{e}_s\rangle_L - \langle \boldsymbol{e}_j, d\boldsymbol{e}_s\rangle_L = -\alpha_s^j.$$

Differentiating (5.6), it holds that

$$0 = \frac{1}{c}\, dd\boldsymbol{e}_0 = \sum_j (d\omega^j \boldsymbol{e}_j - \omega^j \wedge d\boldsymbol{e}_j) = \sum_{j,s} \omega^s \wedge \omega_s^j \boldsymbol{e}_j - \sum_{j,s} \omega^j \wedge \alpha_j^s \boldsymbol{e}_s = \sum_j \sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j)\boldsymbol{e}_j$$

because $\omega^j \wedge \omega^j = 0$. Thus, we have $\sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j) = 0$, and then

$$\begin{aligned}
0 &= \left(\sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j)\right)(\boldsymbol{e}_l, \boldsymbol{e}_m) = (\omega_l^j(\boldsymbol{e}_m) - \alpha_l^j(\boldsymbol{e}_m)) - (\omega_m^j(\boldsymbol{e}_l) - \alpha_m^j(\boldsymbol{e}_l)),\\
0 &= (\omega_j^m(\boldsymbol{e}_l) - \alpha_j^m(\boldsymbol{e}_l)) - (\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)) = -(\omega_m^j(\boldsymbol{e}_l) - \alpha_m^j(\boldsymbol{e}_l)) - (\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)),\\
0 &= (\omega_m^l(\boldsymbol{e}_j) - \alpha_m^l(\boldsymbol{e}_j)) - (\omega_j^l(\boldsymbol{e}_m) - \alpha_j^l(\boldsymbol{e}_m)) = -(\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)) + (\omega_l^j(\boldsymbol{e}_m) - \alpha_l^j(\boldsymbol{e}_m)),
\end{aligned}$$

which conclude that $\omega_l^j = \alpha_l^j$. Summing up, we have

$$d\boldsymbol{e}_j = c\omega^j \boldsymbol{e}_0 + \sum_s \omega_j^s \boldsymbol{e}_s. \tag{5.7}$$

Then the frame $\mathcal{F}$ in (5.5) satisfies

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \text{where} \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & c\,{}^t\boldsymbol{\omega} \\ c\boldsymbol{\omega} & \Omega \end{pmatrix} \quad \text{and} \quad \boldsymbol{\omega} := {}^t(\omega^1, \dots, \omega^n). \tag{5.8}$$

The integrability condition of (5.8) is

$$O = d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega} = \begin{pmatrix} c^2\,{}^t\boldsymbol{\omega} \wedge \boldsymbol{\omega} & c\,(d\,{}^t\boldsymbol{\omega} + {}^t\omega \wedge \Omega) \\ c\,(d\boldsymbol{\omega} + \Omega \wedge \boldsymbol{\omega}) & d\Omega + \Omega \wedge \Omega + c^2 \boldsymbol{\omega} \wedge {}^t\boldsymbol{\omega} \end{pmatrix}.$$

The lower-right components of the identity above yields

$$\kappa_i^j + c^2 \omega^i \wedge \omega^j = 0.$$

Hence the sectional curvature of $(H^n(-c^2), g_H) = -c^2$. □

Remark 5.6. One can show the completeness of $(H^n(-c^2), g_H)$. Hence the hyperbolic space is a simply connected space form of negative sectional curvature.

Exercises

5-1 Prove that the sphere

$$S^n(c^2) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = \frac{1}{c^2} \right\}$$

of radius $1/c$ in the Eucidean $n+1$-space is of constant sectional curvature c^2.

5-2 Let$f\colon U \to \mathbb{R}^{n+1}$ be an immersion defined on a domain $U \subset \mathbb{R}^n$, and ν a unit normal vector field. Take an orthornormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ of the tangent bundle of U, and consider each $\boldsymbol{e}_j$ a map into $\mathbb{R}^{n+1}$. In addition, we consider ν an $\mathbb{R}^{n+1}$-valued function. Prove that

$$d\nu = -\sum_j h^j \boldsymbol{e}_j, \qquad \text{where} \qquad h^j := \langle d\boldsymbol{e}_j, \nu\rangle.$$