6 Local uniqueness of space forms

6.1 *Isometries*

A C^∞-map $f\colon M \to N$ between manifolds M and N induces a linear map

$$(df)_p\colon T_pM \ni X \longmapsto (df)_p(X) = \left.\frac{d}{dt}\right|_{t=0} f\circ\gamma(t) \in T_{f(p)}N,$$

where $\gamma\colon(-\varepsilon,\varepsilon)\to M$ is a smooth curve with $\gamma(0)=p$ and $\dot\gamma(0)=X$, called the *differential* of f. Since $p\in M$ is arbitrary, this induces a bundle homomorphism $df\colon TM\to TN$.

Definition 6.1. A *vector field on N along a smooth map $f\colon M\to N$* is a map $X\colon M\to TN$ satisfying $\pi\circ X = f$, where $\pi\colon TN\to N$ is the canonical projection.

Then for each vector field $X\in\mathfrak{X}(M)$, $df(X)$ is a vector field on N along f.

Definition 6.2. A C^∞-map $f\colon M\to N$ between Riemannian manifolds (M,g) and (N,h) is called a *local isometry* if $\dim M = \dim N$ and $f^*h = g$ hold, that is,

$$f^*h(X,Y) := h(df(X), df(Y)) = g(X,Y)$$

holds for X, $Y\in T_pM$ and $p\in M$.

Lemma 6.3. *A local isometry is an immersion.*

Proof. Let $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ be a (local) orthonormal frame of M, where $n=\dim M$. Set $\boldsymbol{v}_j := df(\boldsymbol{e}_j)$ $(j=1,\dots,n)$ for a smooth map $f\colon(M,g)\to(N,h)$. If f is a local isometry, $[\boldsymbol{v}_1(p),\dots,\boldsymbol{v}_n(p)]$ is an orthonormal system in $T_{f(p)}N$, because

$$h(\boldsymbol{v}_i,\boldsymbol{v}_j) = h(df(\boldsymbol{e}_i), df(\boldsymbol{e}_j)) = f^*h(\boldsymbol{e}_i,\boldsymbol{e}_j) = g(\boldsymbol{e}_i,\boldsymbol{e}_j).$$

Hence the differential $(df)_p$ is of rank n. □

The proof of Lemma 6.3 suggests the following fact:

Corollary 6.4. *A smooth map $f\colon(M,g)\to(N,h)$ is a local isometry if and only if for each $p\in M$,*

$$[\boldsymbol{v}_1,\dots,\boldsymbol{v}_n] := [df(\boldsymbol{e}_1),\dots,df(\boldsymbol{e}_n)]$$

is an orthonormal frame for some orthonormal frame $[\boldsymbol{e}_j]$ on a neighborhood of p.

6.2 *Local uniqueness of space forms*

Theorem 6.5. *Let $U\subset\mathbb{R}^n$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (U,g) is constant k, there exists a local isometry $f\colon U\to N^n(k)$, where*

$$N^n(k) = \begin{cases} S^n(k) & (k>0) \\ \mathbb{R}^n & (k=0) \\ H^n(k) & (k<0). \end{cases}$$

Proof. Take an orthonormal frame $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ on U, and let (ω^j), $\Omega=(\omega_i^j)$ and $K=(\kappa_i^j)$ be the dual frame, the connection form, and the curvature form with respect to $[\boldsymbol{e}_j]$, respectively. Since the sectional curvature is constant k, $\kappa_i^j = k\omega^i\wedge\omega^j$ holds for each (i,j), because of Theorem 5.1.

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First, consider the case $k = 0$: In this case, $K = d\Omega + \Omega \wedge \Omega = O$, and then by Theorem 3.5, there exists the unique matrix valued function $\mathcal{F}\colon U \to \mathrm{SO}(n)$ satisfying

$$d\mathcal{F} = \mathcal{F}\Omega, \qquad \mathcal{F}(p_0) = \mathrm{id},$$

where $p_0 \in U$ is a fixed point. Decompose the matrix $\mathcal{F}$ into column vectors as $\mathcal{F} = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$, and define an $\mathbb{R}^n$-valued one form

$$\boldsymbol{\alpha} := \sum_{j=1}^{n} \omega^j \boldsymbol{v}_j.$$

Then

$$d\boldsymbol{\alpha} = \sum_{j=1}^{n}\left(d\omega^j \boldsymbol{v}_j - \omega^j \wedge d\boldsymbol{v}_j\right) = \sum_{j,s}\left(\omega^s \wedge \omega_s^j\right)\boldsymbol{v}_j - \sum_{j,s}\left(\omega^j \wedge \omega_j^s\right)\boldsymbol{v}_s = \boldsymbol{0}.$$

Hence by the Poincaré lemma (Theorem 3.8), there exists a smooth map $f\colon U \to \mathbb{R}^n$ satisfying $df = \boldsymbol{\alpha}$. For such an f, it holds that

$$df(\boldsymbol{e}_s) = \alpha(\boldsymbol{e}_s) = \sum_{j=1}^{n} \omega^j(\boldsymbol{e}_s)\boldsymbol{v}_j = \boldsymbol{v}_s$$

for $s = 1, \dots, n$. Hence $[df(\boldsymbol{e}_1), \dots, df(\boldsymbol{e}_n)] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ is an orthonormal frame, and then f is a local isometry because Corollary 6.4.

Next, consider the case $k = -c^2 < 0$. We set

$$\widetilde{\Omega} := \begin{pmatrix} 0 & c\,{}^t\boldsymbol{\omega} \\ c\boldsymbol{\omega} & \Omega \end{pmatrix}, \qquad \text{where} \qquad \boldsymbol{\omega} = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}$$

as in (5.8) in Section 5[7]. Since $\kappa_i^j = k\omega^i \wedge \omega^j = -c^2\omega^i \wedge \omega^j$, $d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega} = O$ holds as seen in Section 5. Hence there exists an matrix valued function $\mathcal{F}\colon U \to \mathrm{M}_{n+1}(\mathbb{R})$ satisfying

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \mathcal{F}(p_0) = \mathrm{id}, \tag{6.1}$$

where $p_0 \in U$ is a fixed point. Notice that

$${}^t\widetilde{\Omega}Y + Y\widetilde{\Omega} = O \qquad Y = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

holds,

$$d(\mathcal{F}Y\,{}^t\mathcal{F}) = \mathcal{F}\widetilde{\Omega}Y\,{}^t\mathcal{F} + \mathcal{F}Y\,{}^t\widetilde{\Omega}\,{}^t\mathcal{F} = \mathcal{F}(\widetilde{\Omega}Y + Y\,{}^t\widetilde{\Omega})\,{}^t\mathcal{F} = O.$$

Hence, by the initial condition,

$$\mathcal{F}Y\,{}^t\mathcal{F} = Y, \qquad \text{that is,} \qquad (\mathcal{F}Y)^{-1} = {}^t\mathcal{F}Y.$$

Thus, we have

$${}^t\mathcal{F}Y\mathcal{F} = (\mathcal{F}Y)^{-1}\mathcal{F} = Y\mathcal{F}^{-1}\mathcal{F} = Y. \tag{6.2}$$

[7]The original version of (5.8) is wrong. See the revised version on July 26.

Decompose $\mathcal{F} = [\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$. Then (6.2) is equivalent to

$$-\langle \boldsymbol{v}_0, \boldsymbol{v}_0\rangle_L = \langle \boldsymbol{v}_1, \boldsymbol{v}_1\rangle_L = \dots = \langle \boldsymbol{v}_n, \boldsymbol{v}_n\rangle_L = 1, \qquad \langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = 0 \quad (\text{if } i \neq j). \tag{6.3}$$

In particular, the 0-th component of $\boldsymbol{v}_0$ never vanishes, since

$$-1 = \langle \boldsymbol{v}_0, \boldsymbol{v}_0\rangle_L = -(v_0^0)^2 + (v_0^1)^2 + \dots + (v_0^n)^2 \qquad \boldsymbol{v}_0 = {}^t(v_0^0, v_0^1, \dots, v_0^n).$$

Moreover, by the initial condition $\boldsymbol{v}_0(p_0) = {}^t(1, 0, \dots, 0)$,

$$v_0^0 > 0 \tag{6.4}$$

holds.

Set $f := \frac{1}{c}\boldsymbol{v}_0$. Then $f\colon U \to \mathbb{R}^{n+1}_1$ is the desired map. In fact, by (6.3) and (6.4),

$$f \in H^n(-c^2) = \left\{ \boldsymbol{x} = {}^t(x^0, \dots, x^n) \in \mathbb{R}^{n+1}_1 \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -\frac{1}{c^2}, cx^0 > 0 \right\},$$

and

$$df(\boldsymbol{e}_j) = \frac{1}{c} d\boldsymbol{v}_0(\boldsymbol{e}_j) = \sum_{s=1}^n \omega^s(\boldsymbol{e}_j)\boldsymbol{v}_s = \boldsymbol{v}_j.$$

Hence $[\boldsymbol{v}_j] = [\boldsymbol{e}_j]$ is an orthonormal frame because (6.3).

The case $k > 0$ is left as an exercise. □

6.3 The fundamental theorem for surfaces revisited

From now on, we restrict our attention to surfaces in 3-dimensional space form. Before stating the fundamental theorem for surfaces in space forms, we review the fundamental theorem of surface theory in the Euclidean 3-space.

Let $f\colon U \to \mathbb{R}^3$ be an immersion of a domain $U \subset \mathbb{R}^2$ into the Euclidean 3-space. The *first fundamental form* ds^2 is the pull-back of the Euclidean metric by f, that is,

$$ds^2(X, Y) = \langle df(X), df(Y)\rangle$$

for all tangent vectors X, Y in T_pU. Since f is an immersion, ds^2 gives an Riemannian metric on U. Take an orthonormal frame $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ on U with respect to ds^2 and denote by (ω^1, ω^2) the dual $[\boldsymbol{e}_1, \boldsymbol{e}_2]$. Since the connection form $\Omega = (\omega_i^j)$ is skew-symmetric, we can write

$$\Omega = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix},$$

where $\mu = \omega_2^1$ is a 1-form on U, and the curvature form is

$$K = d\Omega + \Omega \wedge \Omega = \begin{pmatrix} 0 & d\mu \\ -d\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \omega^1 \wedge \omega^2$$

where k is the sectional curvature of ds^2.

Set

$$\boldsymbol{v}_j := df(\boldsymbol{e}_j) \quad (j = 1, 2) \qquad \text{and} \qquad \boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2,$$

where "×" denotes the vector product of $\mathbb{R}^3$. Then $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ is orthonormal in $\mathbb{R}^3$. In particular we have the orthogonal matrix valued function

$$\mathcal{F}\colon U \ni p \mapsto \mathcal{F}(p) = [\boldsymbol{v}_1(p), \boldsymbol{v}_2(p), \boldsymbol{v}_3(p)] \in \mathrm{SO}(3),$$

which is called the *adapted frame* of f with respect to $[\boldsymbol{e}_j]$. We call $\boldsymbol{v}_3$ the *unit normal vector field* of f. Define two differential forms h^j by

$$h^j := -\langle d\boldsymbol{v}_3, \boldsymbol{v}_j\rangle \qquad (j = 1, 2). \tag{6.5}$$

The pair $(h^j)_{j=1,2}$ is called the *second fundamental form*.

Lemma 6.6.

$$\begin{aligned} d\boldsymbol{v}_1 &= -\mu \boldsymbol{v}_2 + h^1 \boldsymbol{v}_3, \\ d\boldsymbol{v}_2 &= \mu \boldsymbol{v}_1 + h^2 \boldsymbol{v}_3, \\ d\boldsymbol{v}_3 &= -h^1 \boldsymbol{v}_1 - h^2 \boldsymbol{v}_2, \end{aligned}$$

in other words,

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & -\mu & -h^1 \\ \mu & 0 & -h^2 \\ h^1 & h^2 & 0 \end{pmatrix}.$$

Exercises

6-1 Prove Theorem 6.5 for $k > 0$.

6-2 Prove Lemma 6.6.