7 Fundamental theorem for surfaces in space forms

7.1 *Surfaces in* 3*-dimensional Riemannian manifolds*

Let (N^3, g) be an oriented Riemannian 3-manifold, and $f\colon M^2 \to N^3$ an immersion of an oriented 2-manifold M^2 into N^3. Then the pull-back ds^2 of the Riemannian metric g by f defined as

$$ds^2(X,Y) := f^*g(X,Y) = g\bigl(df(X), df(Y)\bigr) \tag{7.1}$$

is a Riemannian metric on M^2. We call ds^2 the *first fundamental form* of f.

Take an orthonormal frame $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ with respect to ds^2 on a domain $U \subset M^2$ compatible to the orientation of M^2, and set $\boldsymbol{v}_j := df(\boldsymbol{e}_j)$ $(j = 1, 2)$. Then at each point $p \in U$, $[\boldsymbol{v}_1(p), \boldsymbol{v}_2(p)]$ is an orthonormal basis of $df(T_pM^2) \subset T_{f(p)}N^3$, by the definition of ds^2:

$$g(\boldsymbol{v}_i, \boldsymbol{v}_j) = g\bigl(df(\boldsymbol{e}_i), df(\boldsymbol{e}_j)\bigr) = ds^2(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij},$$

where δ_{ij} denotes Kronecker's delta symbol.

Since $df(T_pM^2)^\perp$ is one dimensional subspace of $T_{f(p)}N^3$ and N^3 is oriented, there exists the unique vector field $\boldsymbol{v}_3$ on M^3 along an immersion f such that $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ is an orthonormal frame compatible to the orientation of N^3. We call $\boldsymbol{v}_3$ the *unit normal vector field* to f. By definition, the unit normal vector field does not depend on choice of orthonormal frame.

If (N^3, g) is expressed as a submanifold of a (pseudo) Euclidean space $\mathbb{R}^N$, each vector $\boldsymbol{v}_j$ is interpreted as an $\mathbb{R}^N$-valued function on U. So we obtain a couple of 1-forms on U by

$$h^j := -g(d\boldsymbol{v}_3, \boldsymbol{v}_j) \qquad (j = 1, 2), \tag{7.2}$$

where d denotes a derivative of a vector valued function on U. The *second fundamental form*, or the *shape operator* of f is defined as

$$\boldsymbol{h} := h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2. \tag{7.3}$$

It can be easily shown that the definition (7.3) of $\boldsymbol{h}$ does not depend on choice of orthonormal frames. Thus, for each $p \in M^2$, we obtain a linear map

$$\boldsymbol{h}\colon T_pM^2 \ni X \mapsto h^1(X)\boldsymbol{e}_1 + h^2(X)\boldsymbol{e}_2 \in T_pM^2.$$

We set functions h^i_j $(i, j = 1, 2)$ on U by

$$h^i_j := h^i(\boldsymbol{e}_j) = g(d\boldsymbol{v}_3(\boldsymbol{e}_i), \boldsymbol{v}_j) = \langle d\boldsymbol{v}_3(\boldsymbol{e}_i), \boldsymbol{v}_j\rangle, \tag{7.4}$$

where $\langle\ ,\ \rangle$ is the inner product of $\mathbb{R}^N$ whose restriction coincides with g.

Lemma 7.1. $h^1_2 = h^2_1$.

Proof. Since $df([\boldsymbol{e}_1, \boldsymbol{e}_2])$ is perpendicular to $\boldsymbol{v}_3$,

$$\begin{aligned}-h^1_2 &= \langle d\boldsymbol{v}_3(\boldsymbol{e}_2), \boldsymbol{v}_1\rangle = \boldsymbol{e}_2\langle \boldsymbol{v}_3, \boldsymbol{v}_1\rangle - \langle \boldsymbol{v}_3, d\boldsymbol{v}_1(\boldsymbol{e}_2)\rangle = -\langle \boldsymbol{v}_3, \boldsymbol{e}_2(df\boldsymbol{e}_1)\rangle \\ &= -\langle \boldsymbol{v}_3, \boldsymbol{e}_1(df(\boldsymbol{e}_2)) + df([\boldsymbol{e}_1, \boldsymbol{e}_2])\rangle = -\langle \boldsymbol{v}_3, d\boldsymbol{v}_2(\boldsymbol{e}_1)\rangle = \langle d\boldsymbol{v}_3(\boldsymbol{e}_1), \boldsymbol{v}_2\rangle = -h^2_1.\end{aligned}$$ □

Definition 7.2. Under the situation above, the *extrinsic curvature* K_{ext} and the *mean curvature* H of the surface f are defined by

$$K_{\mathrm{ext}} := h^1_1h^2_2 - h^1_2h^2_1 = h^1 \wedge h^2(\boldsymbol{e}_1, \boldsymbol{e}_2), \qquad H := \frac{h_{11} + h_{22}}{2}.$$

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7.2 The fundamental theorem for surfaces in space forms

Let $f\colon M^2 \to N^3(k)$ be an immersion, where k_0 is a real number and

$$N^3(k_0) := \begin{cases} H^3(k_0) & (k_0 < 0), \\ \mathbb{R}^3 & (k_0 = 0), \\ S^3(k_0) & (k_0 > 0). \end{cases}$$

We denote ds^2 and $\boldsymbol{h}$ by the first and second fundamental forms of f. Take a orthonormal frame $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ on a domain $U \subset (M^2, ds^2)$, and denote by (ω^j) its dual frame. Then

$$d\mu = k\omega^1 \wedge \omega^2 \qquad (\mu := \omega^1_2) \tag{7.5}$$

holds, where ω^1_2 is the component of the connection form, and k is the sectional curvature.

Theorem 7.3. *Under the situation above,*

$$k = K_{\mathrm{ext}} + k_0, \qquad dh^1 = h^2 \wedge \mu, \qquad dh^2 = -h^1 \wedge \mu \tag{7.6}$$

hold.

Proof. First we assume $k_0 = 0$, that is, $f\colon M^2 \to \mathbb{R}^3$. We set $\mathcal{F} := (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$, where $\boldsymbol{v}_j = df(\boldsymbol{e}_j)$ $(j = 1, 2)$ and $\boldsymbol{v}_3$ is the unit normal vector field. Then the map $\mathcal{F}\colon U \to \mathrm{SO}(3)$ satisfies

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & \mu & -h^1 \\ -\mu & 0 & -h^2 \\ h^1 & h^2 & 0 \end{pmatrix} \tag{7.7}$$

as seen in Exercise 6-2. Then the compatibility condition for (7.7) is computed as

$$\begin{aligned} O = d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega} &= \begin{pmatrix} 0 & d\mu & -dh^1 \\ -d\mu & 0 & -dh^2 \\ dh^1 & dh^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -h^1 \wedge h^2 & -\mu \wedge h^2 \\ -h^2 \wedge h^1 & 0 & \mu \wedge h^1 \\ -h^2 \wedge \mu & h^1 \wedge \mu & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & d\mu - h^1 \wedge h^2 & -dh^1 + h^2 \wedge \mu \\ -d\mu + h^1 \wedge h^2 & 0 & -dh^2 - h^1 \wedge \mu \\ dh^1 - h^2 \wedge \mu & dh^2 + h^1 \wedge \mu & 0 \end{pmatrix}, \end{aligned}$$

which is equivalent to (7.6) for $k_0 = 0$.

Next, we consider the case $k_0 = -c^2 < 0$. Let $\boldsymbol{v}_j = df(\boldsymbol{e}_j)$ $(j = 1, 2)$ and take the unit normal vector field $\boldsymbol{v}_3$. Setting $\boldsymbol{v}_0 = cf$, $\mathcal{F} := (\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ is a (pseudo) orthonormal frame of $\mathbb{R}^4_1$ along the map f. Here, we have

$$\begin{aligned} d\boldsymbol{v}_0 &= c\,df = c(\omega^1 \boldsymbol{v}_1 + \omega^2 \boldsymbol{v}_2), \\ d\boldsymbol{v}_1 &= c\omega^1 \boldsymbol{v}_0 - \mu \boldsymbol{v}_2 + h^1 \boldsymbol{v}_3, \\ d\boldsymbol{v}_2 &= c\omega^2 \boldsymbol{v}_0 + \mu \boldsymbol{v}_1 + h^2 \boldsymbol{v}_3, \\ d\boldsymbol{v}_3 &= -h^1 \boldsymbol{v}_1 - h^2 \boldsymbol{v}_3, \end{aligned}$$

that is,

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & c\omega^1 & c\omega^2 & 0 \\ c\omega^1 & 0 & \mu & -h^1 \\ c\omega^2 & -\mu & 0 & -h^2 \\ 0 & h^1 & h^2 & 0 \end{pmatrix}. \tag{7.8}$$

The integrability condition of (7.8) is

$$O = d\widetilde{\Omega} + \widetilde{\Omega}\wedge\widetilde{\Omega} = \begin{pmatrix} 0 & c\,d\omega^1 & c\,d\omega^2 & 0 \\ c\,d\omega^1 & 0 & d\mu & -dh^1 \\ c\,d\omega^2 & -d\mu & 0 & -dh^2 \\ 0 & dh^1 & dh^2 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & -c\omega^2\wedge\mu & c\omega^1\wedge\mu & -c(\omega^1\wedge h^1 + \omega^2\wedge h^2) \\ c\mu\wedge\omega^2 & 0 & c^2\omega^1\wedge\omega^2 - h^1\wedge h^2 & -\mu\wedge h^2 \\ -c\mu\wedge\omega^1 & c^2\omega^2\wedge\omega^1 - h^2\wedge h^1 & 0 & \mu\wedge h^1 \\ 0 & -h^2\wedge\mu & h^1\wedge\mu & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d\mu + c^2\omega^1\wedge\omega^2 - h^1\wedge h^2 & -dh^1 + h^2\wedge\mu \\ 0 & -d\mu - c^2\omega^1\wedge\omega^2 + h^1\wedge h^2 & 0 & -dh^2 - h^1\wedge\mu \\ 0 & dh^1 - h^2\wedge\mu & dh^2 + h^1\wedge\mu & 0 \end{pmatrix},$$

here we used the relation

$$d\omega^1 = \sum_s \omega^s\wedge\omega^1_s = \omega^2\wedge\omega^1_2 = \omega^2\wedge\mu, \qquad d\omega^2 = -\omega^1\wedge\mu,$$

and (cf. Lemma 7.1)

$$(\omega^1\wedge h^1 + \omega^2\wedge h^2)(\boldsymbol{e}_1,\boldsymbol{e}_2) = h^1(\boldsymbol{e}_2) - h^2(\boldsymbol{e}_1) = h^1_2 - h^2_1 = 0.$$

Thus, we have (7.6) for $k_0 = -c^2$.

The case $k_0 = c^2 > 0$, the adapted frame $\mathcal{F} := (\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$, where $\boldsymbol{v}_0 = cf$ satisfies

$$(7.9) \qquad d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & -c\omega^1 & -c\omega^2 & 0 \\ c\omega^1 & 0 & \mu & -h^1 \\ c\omega^2 & -\mu & 0 & -h^2 \\ 0 & h^1 & h^2 & 0 \end{pmatrix},$$

whose integrability condition is equivalent to (7.6) for $k_0 = c^2$. □

Since (7.6) is the integrability condition for (7.7), (7.8) or (7.9), the following "fundamental theorem" holds:

Theorem 7.4 (The fundamental theorem for surfaces)**.** *Let $U \subset \mathbb{R}^2$ be a simply connected domain and ds^2 a Riemannian metric on U. Assume that a pair of one forms (h^1, h^2) satisfies* (7.6) *for a real number k_0, where $\mu = \omega^1_2$ is the connection form with respect to an orthonormal frame $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ on U. Then there exists an immersion $f\colon U \to N^3(k_0)$ whose first and second fundamental forms are ds^2 and $\boldsymbol{h} = h^1\boldsymbol{e}_1 + h^2\boldsymbol{e}_2$, respectively.*

The first equality of (7.6) is called the *Gauss equation*, and the last two equalities the *Codazzi equation*. Remark that the Codazzi equation does not depend on the curvature k_0 of the ambient space.

Lemma 7.5. *Let (U, ds^2) be a domain of $\mathbb{R}^2$ with Riemannian metric ds^2. Take an orthonormal frame $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ and denote by $\mu = \omega^1_2$ its connection form. If a pair (h^1, h^2) of one forms satisfy the Codazzi equation*

$$(7.10) \qquad dh^1 = h^2\wedge\mu, \qquad dh^2 = -h^1\wedge\mu,$$

another pair $(\tilde{h}^1, \tilde{h}^2) = (h^1 + t\omega^1, h^2 + t\omega^2)$ also satisfies the Codazzi equation:

$$d\tilde{h}^1 = \tilde{h}^2\wedge\mu, \qquad d\tilde{h}^2 = -\tilde{h}^1\wedge\mu,$$

where (ω^j) is the dual of $[\boldsymbol{e}_j]$.

Proof. By Lemma 2.17, we have

$$d\omega^1 = \omega^2 \wedge \omega^1_2 = \omega^2 \wedge \mu, \qquad d\omega^2 = \omega^1 \wedge \omega^2_1 = -\omega^1 \wedge \mu,$$

namely, (ω^1, ω^2) satisfies the Codazzi equation (7.10). The equation (7.10) is linear in (h^j), the conclusion follows. □

Thus we have the following, so called the "Lawson correspondence" theorem:

Theorem 7.6. *Let $f\colon U \to N^3(k_0)$ be an immersion of constant mean curvature H defined on a simply-connected domain $U \subset \mathbb{R}^2$. Then there exists an immersion $f_{\tilde{k}_0}\colon U \to N^3(\tilde{k}_0)$ of constant mean curvature $H + t$ sharing the first fundamental form with f, where $\tilde{k}_0 = k_0 - t^2 - 2Ht$.*

Proof. Let (h^1, h^2) be the second fundamental form of f. Setting $(\tilde{h}^1, \tilde{h}^2) = (h^1, h^2) + (\omega^1, \omega^2)$, the first fundamental form ds^2 of f and $(\tilde{h}^1, \tilde{h}^2)$ satisfies (7.6) for $\tilde{k}_0$. Thus, we have $f_{\tilde{k}_0}$ as desired. □

Example 7.7. Let $f\colon U \to \mathbb{R}^3$ be a minimal surface (that is, with zero mean curvature). Then there exists $f_1\colon U \to H^3(-1)$ of constant mean curvature 1 with the same first fundamental form as f_1.