Advanced Topics in Geometry E1 (MTH.B505)

Inner products

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Our Goal (of MTH.B505–506)

Theorem (Fundamental thm of Riemannian mfd)

A complete simply connected Riemannian n-manifold of constant sectional curvature k is isometric to

- *the Euclidean space $\mathbb{R}^n$ when $k = 0$,*
- *the n-dimensional sphere $S^n(k) \subset \mathbb{R}^{n+1}$ if $k > 0$, and*
- *the n-dimensional hyperbolic space $H^n(k)$ if $k < 0$.*

cf. The fundamental theorem for surface theory

Our Goal (of MTH.B505–506)

E1 F1

(domain)

$U \subset \mathbb{R}^n$

#506 #505

- a simply-connected Riemannian n-manifold
- complete
- sectional curvature k
- isometric
- the Euclidean space $\mathbb{R}^n$
- the sphere S^n
- the hyperbolic space H^n

· topological property

"no boundary in finite distance"

manifold of dim n

manifold equipped with a Riem. metric

(next week)

· inner products on "Tangent space"

Riemannian manifold

- a Riemannian n-manifold #505

simple connectedness : #506

geodesics 測地線

~ lines

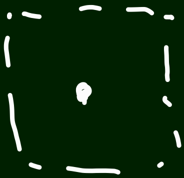
- completeness #505

'no boundaries in finite distance'

(ex) the plane : complete

the plane − {one point}

: incomplete



· standard models of "spaces"

- the Euclidean space $\mathbb{R}^n$

  well-known.

- the sphere $S^n \subset \mathbb{R}^{n+1}$ $\quad S^n = \{ x \mid |x| = 1 \}$

  球面

- the hyperbolic space H^n

  · a stage of "non-euclidean geom"
  · defined on #505
  · space of constant negative curv.

#505

#506

cf. Advanced topics in Geometry F1 (MTH.B506) on 2Q.

- curvature (曲率) · how "curve" the space

☆ integrability condition (可積分条件)

" $F_{xy} = F_{yx}$ " for solvability of a system of partial differential equations

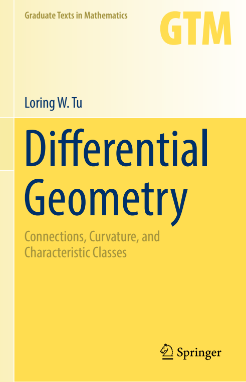
ex
$$\begin{cases} \dfrac{\partial u}{\partial x} = y \\ \dfrac{\partial u}{\partial y} = ax \end{cases}$$

- If $a = 1 \Rightarrow \exists$ sol $\boxed{u = xy}$
- Otherwise $\nexists$ sol

∵ If u: sol

$$\Rightarrow \boxed{\underbrace{\frac{\partial^2 u}{\partial x \partial y}}_{a} \neq \underbrace{\frac{\partial^2 u}{\partial y \partial x}}_{1}}$$

References



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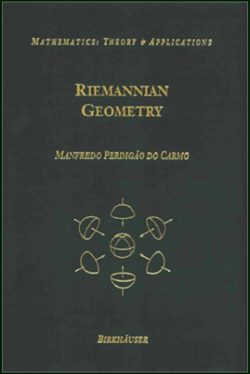

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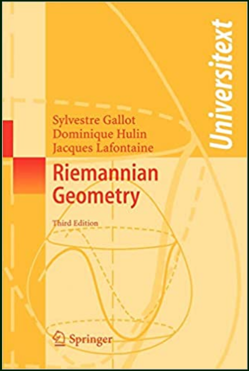

Gallot et. al.

J. Milnor "Morse Theory".