Advanced Topics in Geometry E1 (MTH.B505)

Inner products

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Inner products

$< \infty$

V: an n-dimensional vector space over $\mathbb{R}$

Definition

An inner product on V is a map

$$\langle\ ,\ \rangle : V \times V \ni (\boldsymbol{x}, \boldsymbol{y}) \mapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{R}$$

which is

- bilinear
- symmetric, and
- positive definite.

$\circ\ \langle\ ,\ \rangle : V \times V \to \mathbb{R}$ is bilinear (双線形)

$\Leftrightarrow$ ① $\langle x, \cdot \rangle : V \ni y \mapsto \langle x, y \rangle \in \mathbb{R}$ is linear

$\cdot\ \langle \cdot, x \rangle : V \ni y \mapsto \langle y, x \rangle \in \mathbb{R}$ is linear

$$\Leftrightarrow ① \begin{cases} \langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle \\ \langle x, \alpha y \rangle = \alpha \langle x, y \rangle \end{cases}$$

$$② \begin{cases} \langle y+z, x \rangle = \quad -- \\ \qquad -- \end{cases}$$

$\circ$ symmetric

$\Leftrightarrow \langle x, y \rangle = \langle y, x \rangle$

Positivity $\langle , \rangle$ symm. bilinear form

$\langle , \rangle$ positive definite

$\Leftrightarrow \quad \langle x, x \rangle > 0$ whenever $x \neq 0$

Example $V = \mathbb{R}^2 \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2$$

$\Rightarrow \langle , \rangle$ is an inner product.

(canonical inner product on $\mathbb{R}^2$)

Inner products on $\mathbb{R}^n$

index : superscript

$$\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

Example

- $\mathbb{R}^n$: the set of n-dim. column vectors
- $A = (a_{ij})$: a symmetric $n \times n$-matrix of real components.

Set

$$\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \ni (\boldsymbol{x}, \boldsymbol{y}) \longmapsto {}^t\boldsymbol{x} A \boldsymbol{y} \in \mathbb{R}.$$

${}^t x = x^T$.

transposition

- $\langle\ ,\ \rangle$ is a symmetric bilinear form.
- $\langle\ ,\ \rangle$ is an inner product iff A is positive definite.

pos. def $\Leftrightarrow$ $\langle x, x\rangle > 0$ for $\forall x \neq 0$.

Fact

An arbitrary inner product of $\mathbb{R}^n$ is expressed in this way.

$$\langle x, y \rangle := x^T A y$$

symm

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \qquad y = \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}$$

$$= (x^1 \cdots x^n) \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}$$

$$= \sum_{i,j=1}^{n} a_{ij} x^i y^j$$

- bilinear
- symmetric

$a_{ij} = a_{ji}$

linear homogeneous in x^i, y^j

Positivity of symmetric matrices

$A = (a_{ij})$: an $n \times n$-matrix of real components.

Definition

(A^T) (対称)

- A is symmetric $\Leftrightarrow {}^tA = A \Leftrightarrow a_{ij} = a_{ji}$ for all indices i and j.
- a symmetric matrix A is positive definite $\overset{\text{def}}{\Leftrightarrow} {}^t\boldsymbol{x}A\boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$.

Fact

- *The eigenvalues of a real symmetric matrix are real numbers.*
- *A real symmetric matrix A is positive definite if and only if the eigenvalues of A are all positive.*

cf. Textbook of linear algebra

二次形式 (sect. of "quadratic forms")

- $\langle\, ,\, \rangle$: an inner product on $\mathbb{R}^n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \cdots \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

the canonical basis

$$a_{ij} := \langle e_i, e_j \rangle$$

$$\Rightarrow \langle x, y \rangle = \sum a_{ij} x^i y^j$$

Example

Example

on $\mathbb{R}^2$

$\langle , \rangle$: bilinear symmetric

Let $A := \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := {}^t\boldsymbol{x}A\boldsymbol{y}$ for $a \in \mathbb{R}$

▶ $\langle \ , \ \rangle$ is an inner product if and only if $|a| < 1$

$\lambda, \mu \in \mathbb{R}$

positivity $\Leftrightarrow$ all eigenvalues of $A > 0$

$\underset{(2 \text{ dim})}{\Leftrightarrow}$ $\left(\underset{\lambda\mu}{\det A} > 0 \quad \underset{\lambda+\mu}{\operatorname{tr} A} > 0 \right)$

$\Leftrightarrow \quad 1 - a^2 > 0 \, , \quad 2 > 0 \Leftrightarrow 1 - a^2 > 0$

Orthonormal basis

$(V, \langle\ ,\ \rangle)$: an n-dimensional $\mathbb{R}$-vector space with inner product $\langle\ ,\ \rangle$.

Definition

正規直交基

An orthonormal basis of $(V, \langle\ ,\ \rangle)$ is an n-tuple $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]$ of elements of V such that

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

$\langle e_i, e_i \rangle = 1$

$e_i \perp e_j$

Kronecker's delta

$V = \mathbb{R}^2$, $\langle , \rangle$: the canonical inner product

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 = (x_1\ x_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= x^T y = x^T I y$$

the identity

Set $\begin{cases} v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{cases}$

$\Rightarrow$ $[v_1, v_2]$: an orthonormal basis

Orthonormal basis

Fact (linear algebra)

- *An orthonormal basis of* $(V, \langle\ ,\ \rangle)$ *is a basis of* V.

- *For two orthonormal bases* $[\boldsymbol{e}_j]$ *and* $[\boldsymbol{f}_j]$*, there exists an orthogonal matrix* P *with*

$$[\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n] = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] P.$$

orthogonal matrix
直交行列

Existence of orthonormal basis

Theorem (Gram-Schmidt orthogonalization)

There exists an orthonormal basis for any finite dimensional vector space V over $\mathbb{R}$ with inner product $\langle \, , \, \rangle$,

The dual basis

V: an n-dimensional $\mathbb{R}$-vector space.

$$V^* := \{\alpha\colon V \to \mathbb{R}\,;\, linear\} \qquad \text{(the dual space)}$$

双対空間
そうつい

- $\langle\ ,\ \rangle$: an inner product;
- $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$: an orthonormal basis

Set

$$\omega^j\colon V \ni \boldsymbol{x} \longmapsto \omega^j(\boldsymbol{x}) := \langle \boldsymbol{e}_j, \boldsymbol{x}\rangle \in \mathbb{R} \qquad j = 1, \dots, n$$

V^*

Fact

- $[\omega^1, \dots, \omega^n]$ *is a basis of* V^**, called the dual basis of* $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$.

Expression of inner products

- $(V, \langle\ ,\ \rangle)$: an n-dimensional vector space with inner product $\langle\ ,\ \rangle$
- $[e_1, \dots, e_n]$: an orthonormal basis.
- $[\omega^1, \dots, \omega^n]$: the dual of $[e_j]$; $\omega^k = \langle e_k, \cdot\rangle$.

Fact

$$\boxed{\langle\ ,\ \rangle = \sum_{j=1}^{n} (\omega^j)^2.}$$

used later

$$\langle x, y\rangle = \sum_{j=1}^{n} \omega^j(x)\,\omega^j(y)$$

$$\langle x, y \rangle = \sum_{j=1}^{n} \omega^j(x)\,\omega^j(y)$$

$\because$ $x = x^1\theta_1 + \cdots + x^n\theta_n$.

$y = y^1\theta_1 + \cdots + y^n\theta_n$.

$\Rightarrow$ $x^j = \omega^j(x)$, $y^j = \omega^j(y)$

$\therefore \langle x, y \rangle = x^1y^1 + \cdots + x^ny^n$,

The Euclidean vector space

- $\mathbb{R}^n$: the vector space consisting of n-dim. column vectors
- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := {}^t\boldsymbol{x}\boldsymbol{y} = \sum_{j=1}^{n} x^j y^j$: The canonical inner product,

  $(\boldsymbol{x} = {}^t(x^1, \dots, x^n)$, $\boldsymbol{y} = {}^t(y^1, \dots, y^n))$.

Definition

$\mathbb{E}^n := (\mathbb{R}^n, \langle\ ,\ \rangle)$: the Euclidean vector space.

The Lorentz-Minkowski vector space ⇒ definition of Hyperbolic space

- $\mathbb{R}^{n+1}$: the vector space of $(n+1)$-dim. column vectors
- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_L := -x^0 y^0 + \sum_{j=1}^{n} x^j y^j$:

$$(\boldsymbol{x} = {}^t(x^0, x^1, \ldots, x^n),\ \boldsymbol{y} = {}^t(y^0, y^1, \ldots, y^n)).$$

The canonical Lorentzian "inner product".

Definition

$\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$: the Lorentz-Minkowski vector space.

(**Note** $\langle x, y \rangle_L = 0$ for $\forall y \in \mathbb{R}^{n+1}$ (fix x)

$\Rightarrow x = 0$ non degeneracy)

Exercise 1-1

Problem (Ex. 1-1)

Let $\langle\ ,\ \rangle$ *be an inner product of* $\mathbb{R}^2$ *defined by*

$$\underline{\langle \boldsymbol{x}, \boldsymbol{y} \rangle := {}^t\boldsymbol{x} A \boldsymbol{y}} \qquad \underline{A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}},$$

where a *is a real number with* $|a| < 1$*.*

- *Find an orthonormal basis* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *with respect to* $\langle\ ,\ \rangle$*.*
- *Find row vectors* $\hat{\omega}^j$ $(j = 1, 2)$ *such that the dual basis* $[\underline{\omega^j}]$ *of* $[\boldsymbol{e}_j]$ *is expressed as*

$\omega^j : \mathbb{R}^2 \to \mathbb{R}$ linear

$$\omega^j(\boldsymbol{x}) = \hat{\omega}^j \boldsymbol{x} \qquad (j = 1, 2)$$

$(1,2)$-matrix

Exercise 1-2

$\langle x, y \rangle_L = (-)x^0y^0 + x^1y^1 + x^2y^2$

Problem (Ex. 1-2)

Let $\mathbb{L}^3$ be the 3-dimensional Lorentz-Minkowski vector space, and fix $\boldsymbol{x} \in \mathbb{L}^3$ with $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -1$. Take the "orthogonal complement"

$$W := \boldsymbol{x}^{\perp} = \{\boldsymbol{y} \in \mathbb{L}^3\,;\, \langle \boldsymbol{x}, \boldsymbol{y} \rangle\}.$$

$\langle x, y \rangle_L = 0$

- *Show that W is ~~an~~ a 2-dimensional linear subspace of $\mathbb{L}^3$.*
- *Show that the restriction of $\langle\ ,\ \rangle_L$ to $W \times W$ is a (positive definite) inner product of W.*

$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \boxed{\langle x, x \rangle_L = -1}$

$W = \left\{ \begin{pmatrix} y^0 \\ y^1 \\ y^2 \end{pmatrix} ; -y^0 x^0 + y^1 x^1 + y^2 x^2 = 0 \right\}$

If $y \in W \rightarrow \boxed{\langle y, y \rangle_L > 0}$ if $y \neq 0$

$(\Leftrightarrow \boxed{\langle x, y \rangle_L = 0})$

conclusion