Advanced Topics in Geometry E1 (MTH.B505)

Inner products

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Inner products

V: an n-dimensional vector space over $\mathbb{R}$

Definition

An inner product on V is a map

$$\langle\ ,\ \rangle : V \times V \ni (\boldsymbol{x}, \boldsymbol{y}) \mapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{R}$$

which is

- bilinear
- symmetric, and
- positive definite.

Inner products on $\mathbb{R}^n$

Example

- $\mathbb{R}^n$: the set of n-dim. column vectors
- $A = (a_{ij})$: a symmetric $n \times n$-matrix of real components.

Set

$$\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \ni (\boldsymbol{x}, \boldsymbol{y}) \longmapsto \boldsymbol{x}^T A \boldsymbol{y} \in \mathbb{R}.$$

- $\langle\ ,\ \rangle$ is a symmetric bilinear form.
- $\langle\ ,\ \rangle$ is an inner product iff A is positive definite.

Fact

An arbitrary inner product of $\mathbb{R}^n$ is expressed in this way.

Positivity of symmetric matrices

$A = (a_{ij})$: an $n \times n$-matrix of real components.

Definition

- A is symmetric $\Leftrightarrow A^T = A \Leftrightarrow a_{ij} = a_{ji}$ for all indices i and j.
- a symmetric matrix A is positive definite $\Leftrightarrow \boldsymbol{x}^T A \boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

Fact

- *The eigenvalues of a real symmetric matrix are real numbers.*
- *A real symmetric matrix A is positive definite if and only if the eigenvalues of A are all positive.*

Example

Example

Let $A := \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^T A \boldsymbol{y}$ for $a \in \mathbb{R}$

- $\langle\ ,\ \rangle$ is an inner product if and only if $|a| < 1$.

Orthonormal basis

$(V, \langle\ ,\ \rangle)$: an n-dimensional $\mathbb{R}$-vector space with inner product $\langle\ ,\ \rangle$.

Definition

An orthonormal basis of $(V, \langle\ ,\ \rangle)$ is an n-tuple $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ of elements of V

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Orthonormal basis

Fact

- *An orthonormal basis of* $(V, \langle\ ,\ \rangle)$ *is a basis of* V.

- *For two orthonormal bases* $[\boldsymbol{e}_j]$ *and* $[\boldsymbol{f}_j]$*, there exists an orthogonal matrix* P *with*

$$[\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n] = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]P.$$

Existence of orthonormal basis

Theorem

There exists an orthonormal basis for any finite dimensional vector space V *over* $\mathbb{R}$ *with inner product* $\langle\ ,\ \rangle$*,*

The dual basis

V: an n-dimensional $\mathbb{R}$-vector space.

$$V^* := \{\alpha\colon V \to \mathbb{R}\,;\, linear\} \qquad \text{(the dual space)}$$

$\langle\ ,\ \rangle$: an inner product;
$[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$: an orthonormal basis
Set

$$\omega^j\colon V \ni \boldsymbol{x} \longmapsto \omega^j(\boldsymbol{x}) := \langle \boldsymbol{e}_j, \boldsymbol{x}\rangle \in \mathbb{R}.$$

Fact

$[\omega^1, \dots, \omega^n]$ *is a basis of* V^**, called the* *dual basis* *of* $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$.

Expression of inner products

- $(V, \langle\ ,\ \rangle)$: an n-dimensional vector space with inner product $\langle\ ,\ \rangle$
- $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$: an orthonormal basis.
- $[\omega^1, \dots, \omega^n]$: the dual of $[\boldsymbol{e}_j]$; $\omega^k = \langle \boldsymbol{e}_k, \cdot \rangle$.

Fact

$$\langle\ ,\ \rangle = \sum_{j=1}^{n} (\omega^j)^2.$$

The Euclidean vector space

- $\mathbb{R}^n$: the vector space consisting of n-dim. column vectors
- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^T \boldsymbol{y} = \sum_{j=1}^{n} x^j y^j$: The canonical inner product,

$$(\boldsymbol{x} = (x^1, \ldots, x^n)^T,\ \boldsymbol{y} = (y^1, \ldots, y^n)^T).$$

Definition

$\mathbb{E}^n := (\mathbb{R}^n, \langle\ ,\ \rangle)$: the Euclidean vector space.

The Lorentz-Minkowski vector space

- $\mathbb{R}^{n+1}$: the vector space of $(n+1)$-dim. column vectors
- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_L := -x^0 y^0 + \sum_{j=1}^{n} x^j y^j$:

  $(\boldsymbol{x} = (x^0, x^1, \ldots, x^n)^T,\ \boldsymbol{y} = (y^0, y^1, \ldots, y^n)^T)$.

  The canonical Lorentzian "inner product".

Definition

$\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$: the Lorentz-Minkowski vector space.

Exercise 1-1

Problem (Ex. 1-1)

Let $\langle\ ,\ \rangle$ *be an inner product of* $\mathbb{R}^2$ *defined by*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^T A \boldsymbol{y} \qquad A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix},$$

where a *is a real number with* $|a| < 1$*.*

- *Find an orthonormal basis* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *with respect to* $\langle\ ,\ \rangle$*.*
- *Find row vectors* $\hat{\omega}^j$ $(j = 1, 2)$ *such that the dual basis* $[\omega^j]$ *of* $[\boldsymbol{e}_j]$ *is expressed as*
$$\omega^j(\boldsymbol{x}) = \hat{\omega}^j \boldsymbol{x} \qquad (j = 1, 2)$$

Exercise 1-2

Problem (Ex. 1-2)

Let $\mathbb{L}^3$ *be the* 3*-dimensional Lorentz-Minkowski vector space, and fix* $\boldsymbol{x} \in \mathbb{L}^3$ *with* $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -1$*. Take the "orthogonal complement"*

$$W := \boldsymbol{x}^{\perp} = \{\boldsymbol{y} \in \mathbb{L}^3 \, ; \, \langle \boldsymbol{x}, \boldsymbol{y} \rangle_L = 0\}.$$

- *Show that* W *is an* 2*-dimensional linear subspace of* $\mathbb{L}^3$*.*
- *Show that the restriction of* $\langle \ , \ \rangle_L$ *to* $W \times W$ *is a* $($*positive definite*$)$ *inner product of* W*.*