Advanced Topics in Geometry E1 (MTH.B505)

Riemannian manifolds

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Exercise 1-1 $\operatorname{tr} A = \lambda_1 + \lambda_2$ $\det A = \lambda_1\lambda_2$ $\lambda_j \in \mathbb{R}$

positive definite

$\Leftrightarrow$ $\forall$ eigenval's of A are positive

Problem (Ex. 1-1)

Let $\langle\ ,\ \rangle$ be an inner product of $\mathbb{R}^2$ defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^T A \boldsymbol{y} \qquad A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix},$$

2-dim $\Leftrightarrow \left(\begin{array}{l} \det A > 0 \\ \operatorname{tr} A > 0 \end{array} \right)$

where a is a real number with $|a| < 1$.

- *Find an orthonormal basis $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ with respect to $\langle\ ,\ \rangle$.*
- *Find row vectors $\hat{\omega}^j$ $(j = 1, 2)$ such that the dual basis $[\omega^j]$ of $[\boldsymbol{e}_j]$ is expressed as*

$$\omega^j(\boldsymbol{x}) = \hat{\omega}^j \boldsymbol{x} \qquad (j = 1, 2)$$

$[e_1 \ e_2] \quad \langle e_i, e_j \rangle = \delta_{ij}$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$ Gram-Schmidt orthogonalization

$\langle v_1, v_1 \rangle = (1, 0)\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \quad v_1$: unit.

$e_1 = v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\langle e_1, v_2 \rangle = (1, 0)\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a$

$\tilde{e}_2 = v_2 - \langle v_2, e_1 \rangle e_1 \quad \perp \ e_1$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - a\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -a \\ 0 \end{pmatrix}$

$$\langle \tilde{e}_2, \tilde{e}_2 \rangle = (-a \;\; 1) \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} -a \\ 1 \end{pmatrix}$$

$\tilde{e}_2 \perp e_1$

$$= (0, 1-a^2) \begin{pmatrix} -a \\ 1 \end{pmatrix} = 1 - a^2 \; (>0)$$

$|a| < 1$

$$e_2 := \frac{1}{\sqrt{1-a^2}} \tilde{e}_2 = \frac{1}{\sqrt{1-a^2}} \begin{pmatrix} -a \\ 1 \end{pmatrix}$$

- $[e_1 \;\; e_2] = \begin{bmatrix} 1 & \frac{-a}{\sqrt{1-a^2}} \\ 0 & 1 \end{bmatrix}$: orthogonal basis of $(\mathbb{R}^2, \langle , \rangle)$

Another orthonormal basis $(|a|<1)$ $a = \cos\theta$

$$[f_1 \;\; f_2] = \frac{1}{2}\begin{bmatrix} \sec\frac{\theta}{2} & -\csc\frac{\theta}{2} \\ \sec\frac{\theta}{2} & \csc\frac{\theta}{2} \end{bmatrix}$$

$(\theta \in (0, \pi))$

(check it!)

$$[e_1, e_2] = \begin{bmatrix} 1 & -a/\sqrt{1-a^2} \\ 0 & 1/\sqrt{1-a^2} \end{bmatrix}$$

$$\omega^1(x) := \langle e_1, x \rangle = (1, 0) \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \underbrace{(1, a)}_{\hat{\omega}^1} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\omega^2(x) := \langle e_2, x \rangle = \frac{1}{\sqrt{1-a^2}} (-a, 1) \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{1-a^2}} \left(0, 1-a^2\right) \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\left(0, \sqrt{1-a^2}\right)}_{\hat{\omega}^2} \begin{pmatrix} x \\ y \end{pmatrix}$$

Exercise 1-2

Problem (Ex. 1-2)

Let $\mathbb{L}^3$ be the 3-dimensional Lorentz-Minkowski vector space, and fix $\hat{\boldsymbol{x}} \in \mathbb{L}^3$ with $\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{x}} \rangle_L = -1$. Take the "orthogonal complement"

$$W := \hat{\boldsymbol{x}}^{\perp} = \{\hat{\boldsymbol{y}} \in \mathbb{L}^3\,;\, \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle_L = 0\}$$

- *Show that W is a 2-dimensional linear subspace of $\mathbb{L}^3$.*
- *Show that the restriction of $\langle\ ,\ \rangle_L$ to $W \times W$ is a (positive definite) inner product of W.*

$\mathbb{L}^3 = (\mathbb{R}^3, \langle\, , \,\rangle_L)$ $\qquad \langle x, y\rangle_L := -x^0y^0 + x^1y^1 + x^2y^2$

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} \quad y = \begin{pmatrix} y^0 \\ y^1 \\ y^2 \end{pmatrix}$$

$$\begin{cases} e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \langle e_0, e_0\rangle_L = -1 < 0 \\ e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \langle e_1, e_1\rangle_L = 1 > 0 \end{cases}$$

"non degenerate" inner product (in an extended sense)

$x \in \mathbb{L}^3 \qquad \langle x, x\rangle_L = -1$

$$W = x^\perp = \{ y \in \mathbb{L}^3 \,;\, \langle x, y\rangle_L = 0 \}$$

- $W \subset \mathbb{L}^3$ 2-dim linear subsp
- $\langle\, , \,\rangle_L \big|_{W \times W}$ is positive def.

$$W = x^{\perp} = \{ y \in \mathbb{L}^3 \,;\, \langle x, y \rangle_L = 0 \} \subset \mathbb{L}^3$$

- Consider a map: $\alpha : \mathbb{L}^3 \ni y \longmapsto \alpha(y) := \langle x, y \rangle_L \in \mathbb{R}$

α: linear

$W = \operatorname{Ker} \alpha \subset \mathbb{L}^3$: a linear subspace.

$\underset{\text{linear}}{\mathbb{R} \supset} \operatorname{Im} \alpha \ni -1$ $\qquad \alpha(x) = \langle x, x \rangle_L \underset{\text{assumption}}{=} -1$

$\operatorname{Im} \alpha = \mathbb{R}$

$$\dim W = \dim \mathbb{L}^3 - \dim \operatorname{Im} \alpha = 3 - 1 = 2 \qquad \square$$

- $W = x^{\perp}$ $\quad \langle y, y\rangle_L > 0$ for $y \in W \setminus \{0\}$

$x = (x^0, x^1, x^2)^T \qquad y = (y^0, y^1, y^2)^T$

- $-(x^0)^2 + (x^1)^2 + (x^2)^2 = \underline{-1}$ ① $\quad \langle x, x\rangle_L = -1$
- $-x^0y^0 + x^1y^1 + x^2y^2 = 0$ ② $\quad \langle x, y\rangle_L = 0$

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① $x^0 \neq 0$ $\quad \boxed{|x^0| \geq 1}$ ②: $y^0 = \frac{1}{x^0}(x^1y^1 + x^2y^2)$

Cauchy-Sch. ①

$$y_0^2 = \frac{1}{(x^0)^2}\left(x^1y^1 + x^2y^2\right)^2 \leq \frac{1}{(x^0)^2}\left((x^1)^2 + (x^2)^2\right)\left((y_1)^2 + (y_2)^2\right)$$

$$= \frac{(x^0)^2 - 1}{(x^0)^2}(y_1^2 + y_2^2) \leq y_1^2 + y_2^2$$

Q: '=' never hold

$\langle y, y\rangle_L = \underline{-(y^0)^2 + (y^1)^2 + (y^2)^2} > 0$

Proposition

Let $\mathbb{L}^{n+1}$ be the n-dimensional Lorentz-Minkowski vector space, and fix $\boldsymbol{x} \in \mathbb{L}^{n+1}$ with $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -1$. Take the "orthogonal complement"

$$W := \boldsymbol{x}^{\perp} = \{\boldsymbol{y} \in \mathbb{L}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{y} \rangle_L = 0\}.$$

Then

- *W is an n-dimensional linear subspace of $\mathbb{L}^{n+1}$.*
- *The restriction of $\langle\ ,\ \rangle_L$ to $W \times W$ is a positive definite inner product of W.*

proof: same

⇝ definition of hyperbolic space.