Advanced Topics in Geometry E1 (MTH.B505)

Riemannian manifolds

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Exercise 1-1

Problem (Ex. 1-1)

Let $\langle\ ,\ \rangle$ *be an inner product of* $\mathbb{R}^2$ *defined by*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^T A \boldsymbol{y} \qquad A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix},$$

where a *is a real number with* $|a| < 1$.

- *Find an orthonormal basis* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *with respect to* $\langle\ ,\ \rangle$.
- *Find row vectors* $\hat{\omega}^j$ $(j = 1, 2)$ *such that the dual basis* $[\omega^j]$ *of* $[\boldsymbol{e}_j]$ *is expressed as*

$$\omega^j(\boldsymbol{x}) = \hat{\omega}^j \boldsymbol{x} \qquad (j = 1, 2)$$

Exercise 1-2

Problem (Ex. 1-2)

Let $\mathbb{L}^3$ *be the* 3*-dimensional Lorentz-Minkowski vector space, and fix* $\hat{\boldsymbol{x}} \in \mathbb{L}^3$ *with* $\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{x}} \rangle_L = -1$*. Take the "orthogonal complement"*

$$W := \hat{\boldsymbol{x}}^{\perp} = \{\hat{\boldsymbol{y}} \in \mathbb{L}^3 \,;\, \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle\}.$$

- *Show that* W *is a* 2*-dimensional linear subspace of* $\mathbb{L}^3$*.*
- *Show that the restriction of* $\langle\ ,\ \rangle_L$ *to* $W \times W$ *is a* $($*positive definite*$)$ *inner product of* W*.*

Proposition

Let $\mathbb{L}^{n+1}$ *be the* n*-dimensional Lorentz-Minkowski vector space, and fix* $\boldsymbol{x} \in \mathbb{L}^{n+1}$ *with* $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -1$*. Take the "orthogonal complement"*

$$W := \boldsymbol{x}^{\perp} = \{\boldsymbol{y} \in \mathbb{L}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{y} \rangle_L = 0\}.$$

Then

- W *is an* n*-dimensional linear subspace of* $\mathbb{L}^{n+1}$*.*
- *The restriction of* $\langle\ ,\ \rangle_L$ *to* $W \times W$ *is a positive definite inner product of* W*.*