Advanced Topics in Geometry E1 (MTH.B505)

Riemannian manifolds

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2023/04/25

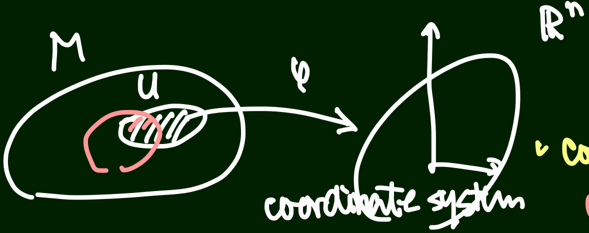
C^∞

A smooth manifold M is

- a topological space (with appropriate properties)
- a family of local coordinate systems (charts)
- smooth coordinate change

Example (The n-dimensional affine space) forget the origin

$\mathbb{R}^n := \{(x^1, \dots, x^n)^T \,;\, x^j \in \mathbb{R}, j = 1, 2, \dots, n\}$



- compatibility

coordinate change : C^∞

$M \subset \mathbb{R}^{n+r}$

manifold

immersion

Fact (Implicit Function Theorem)

- $\boldsymbol{F}\colon \mathbb{R}^{n+r} \to \mathbb{R}^r$: *smooth* C^∞
- $M := \boldsymbol{F}^{-1}(\mathbf{0}) = \{p \in \mathbb{R}^{n+r}\,;\, \boldsymbol{F}(p) = \mathbf{0}\} \neq \emptyset$
- $\operatorname{rank} d\boldsymbol{F} = r$ *on* M

$(n+r) \times r$ matrix

$\Rightarrow$ *M is an n-dimensional submanifold of $\mathbb{R}^{n+r}$*

Example: spheres

$F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

- $n \geqq 1$
- $k > 0$
- $F(\boldsymbol{x}) := F(x^0, \dots, x^n) = \left(\sum_{j=0}^{n} \cancel{j=0}^{n} (x^j)^2\right) - \dfrac{1}{k} = \langle \boldsymbol{x}, \boldsymbol{x}\rangle - \dfrac{1}{k}.$

$$\boxed{S^n(k) = F^{-1}(\{0\}) \subset \mathbb{R}^{n+1}} \leftarrow \text{submanifold (}n\text{-dim)}$$

$$dF = \left(\frac{\partial F}{\partial x^0}, \dots, \frac{\partial F}{\partial x^n}\right) = (2x^0, 2x^1, \dots, 2x^n)$$

$$= 2x$$

$$\operatorname{rank} dF = 1 \iff x \neq 0$$

$0 \notin F^{-1}(0)$

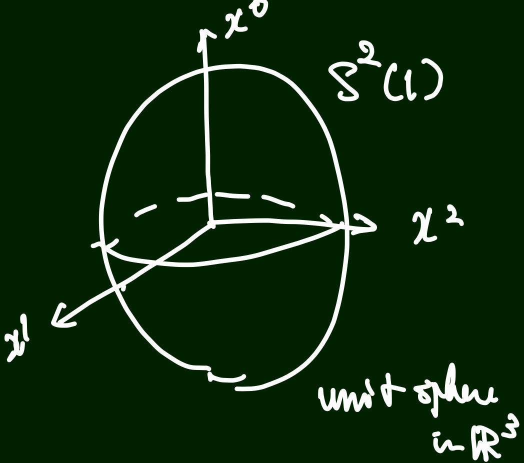
$\because\ F(0) = -\dfrac{1}{k} \neq 0$

$$S^n(k) := \{ x \in \mathbb{R}^{n+1} ; \langle x, x \rangle - \frac{1}{k} = 0 \}$$

- n-dim sphere (of radius $\frac{1}{\sqrt{k}}$)

$n = \cancel{3}\, 2$

$k = 1$



Smooth functions

$\mathcal{F}(M)$: an algebra consists of smooth functions on M.

- $\mathcal{F}(\mathbb{R}^n)$ is the set of smooth functions of n-variables.
- For an n-dimensional submanifold $M \in \mathbb{R}^N$, an element of $\mathcal{F}(M)$ is a function $f\colon M \to \mathbb{R}$ such that $f \circ \varphi$ is of class C^∞ for any smooth function $\varphi\colon \mathbb{R}^n \to M$.

$$\cdot\ \mathcal{F}(M) := \{ f : M \xrightarrow{C^\infty} \mathbb{R} \}$$

$$\left(\begin{array}{lll} f + g & & \alpha f \\ fg & & \end{array} \right)$$

Tangent spaces

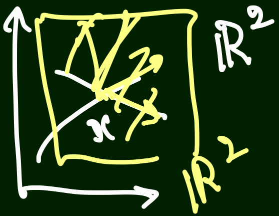
T_pM: the tangent space of a manifold M at $p \in M$.

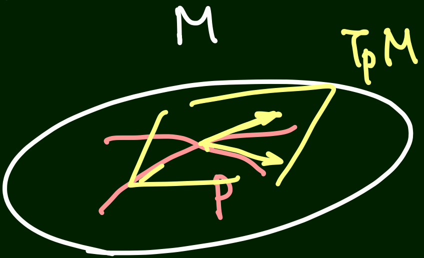
- the set of "velocity vectors" of curves on M passing through p.
- the set of "directional derivatives" at p.

Example $\forall x \in \mathbb{R}^n$

$T_{\boldsymbol{x}}\mathbb{R}^n = \mathbb{R}^n$.

vector space





n-dim

$\{\langle x, x\rangle = \frac{1}{k}\} \subset \mathbb{R}^{n+1}$

Example

$$\underline{T_x S^n(k)} = x^\perp = \boxed{\{v \in \mathbb{R}^{n+1};\ \langle v, x\rangle = 0\}} \subset \mathbb{R}^{n+1}$$

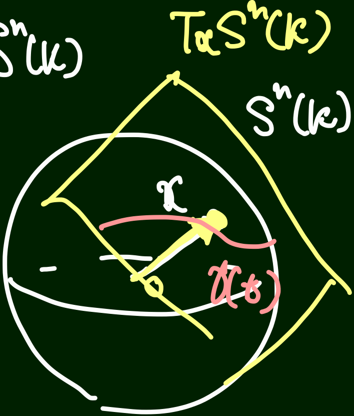
$\cap \,\, \| \quad x^\perp$

For a curve $\gamma(t)$ on $S^n(k)$ with $\gamma(0) = x$

$$\Rightarrow \frac{d\gamma}{dt}(0) \perp x$$

$$\because \quad \frac{d}{dt}\langle \gamma(t), \gamma(t)\rangle = \frac{1}{k}$$

$$\langle \dot\gamma(0), \gamma(0)\rangle = 0$$



 direct sum 接束 $TM \ni v_p \mapsto p = \pi(v_p)$

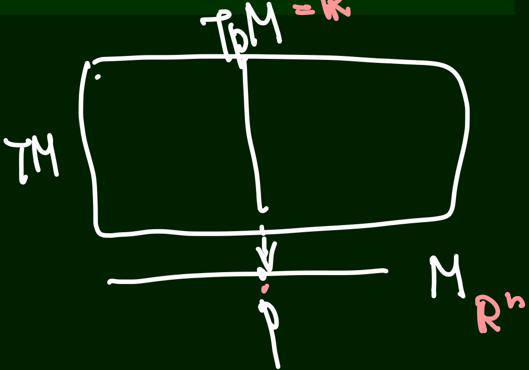
T_pM

$TM = \cup_{p \in M} T_pM$; $\pi\colon TM \to M$: the projection

- "vector bundle of rank $n := \dim M$" over M.
- a $2n$-dimensional manifold.

Example

$T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.



Example

$$TS^n(k) = \cancel{?!} = \{(\boldsymbol{x}, \boldsymbol{v}) \in S^n(k) \times \mathbb{R}^{\cancel{n}\, n+1} ; \langle \boldsymbol{v}, \boldsymbol{x} \rangle = 0\} \cancel{\not\subset} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$

$S^n(k)$ $T_x S^n(k)$

(2n-dimensional submanifold of $\mathbb{R}^{2(n+1)}$)

A vector field X of a manifold M is a smooth map $X : \boxed{M \to TM}$ such that $\pi \circ X = \mathrm{id}$.

$\mathfrak{X}(M)$ = the set of vector fields on M

- $\mathfrak{X}(M)$ is a vector space.
- $\mathfrak{X}(M)$ is an $\mathcal{F}(M)$-module

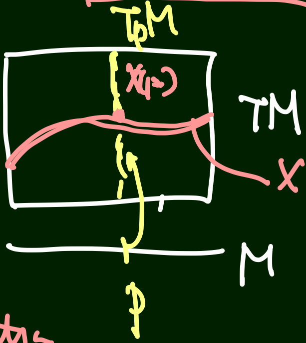
$p \in M$

$\boxed{X(p) \in T_pM}$

set of C^∞-functions

fX

pointwise scalar multiplication



Vector fields on $\mathbb{R}^n$ and $S^n(k)$

- $\mathfrak{X}(\mathbb{R}^n) = \{X : \mathbb{R}^n \to \mathbb{R}^n; C^\infty\}$
- $\mathfrak{X}(S^n(k)) = \{X = (\boldsymbol{x}, \boldsymbol{v}) : S^n(k) \ni \boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^{2n}; \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0\}$

Riemannian manifolds = manifold with Riem. metric

Definition

リーマン計量

A Riemannian metric g on an n-manifold M is a correspondence $p \mapsto g_p$ of p to an inner product g_p of T_pM, which satisfies the smoothness condition, that is,

$\in T_pM$

$$g(X,Y) : M \ni p \mapsto g_p(X_p, Y_p) \in \mathbb{R}$$

is a smooth function for each pair of sooth vector fields X, Y.

Example

✓ $\mathbb{E}^n := (\mathbb{R}^n, \langle\ ,\ \rangle)$ is a Riemannian manifold, called the Euclidean n-space.

canonical inner product of $\mathbb{R}^n = T_x\mathbb{R}^n$

Riemannian submanifold

$M \subset \mathbb{R}^{n+r}$: a submanifold.

- $T_pM \subset T_p\mathbb{R}^{n+r}$.
- The restriction of $\langle\ ,\ \rangle$ to $T_pM \times T_pM$ is an inner product.

$\Rightarrow \langle\ ,\ \rangle$ induces a Riemannian metric on M, called the induced metric.

$= \mathbb{E}^{n+1}$

Example

$S^n(k) \subset \mathbb{R}^{n+1}$ is a Riemannian manifold with the induced metric from $\mathbb{R}^{n+1}$, called the sphere of constant curvature k.

$$T_x S^n(k) = x^\perp \subset \mathbb{R}^{n+1}$$

$\langle , \rangle|_{T_x S^n(k)}$: inner product.

$\rightsquigarrow$ Riemannian metric

The hyperbolic space

Recall: the Lorentz-Minkowski vector space
$\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$.

- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_L := -x^0 y^0 + \sum_{j=1}^{n} x^j y^j$:

  $(\boldsymbol{x} = (x^0, x^1, \dots, x^n)^T,\ \boldsymbol{y} = (y^0, y^1, \dots, y^n)^T)$.

  The canonical Lorentzian "inner product".

For $k < 0$,

$$H^n(k) := \{\boldsymbol{x} = (x^0, \dots, \overset{x^n}{\cancel{n}})^T \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = 1/k, x^0 > 0\}$$

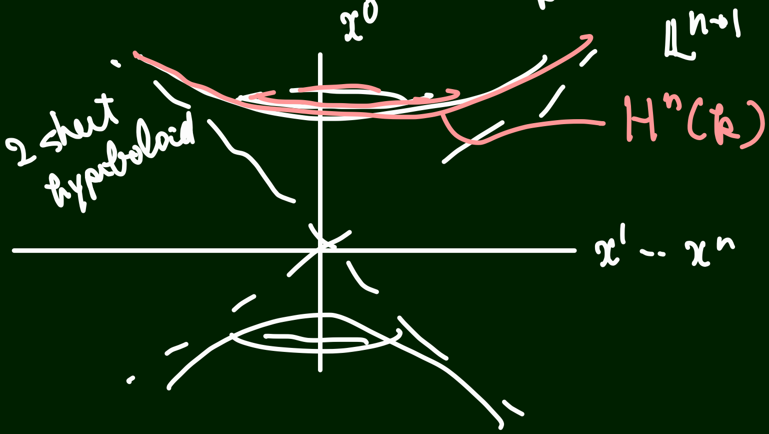
- $T_{\boldsymbol{x}} H^n(k) = \{\boldsymbol{v} \in \mathbb{L}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{v} \rangle\, 0\}$

Fact

The restriction of $\langle\ ,\ \rangle_L$ to $T_{\boldsymbol{x}} H^n(k) \times T_{\boldsymbol{x}} H^n(k)$ is a positive definite inner product, which induces the Riemannian metric $\langle\ ,\ \rangle_L$ to $H^n(k)$. The Riemannian manifold $(H^n(k), \langle\ ,\ \rangle_L)$ is called the

$k < 0 \qquad H^n(k) = \{ x \in \mathbb{L}^{n+1} ; \langle x, x \rangle_1 = \frac{1}{k} \quad x^0 > 0 \}$

$$-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = \frac{1}{k} < 0$$



$$H^n(k) = \{ x = (x^0, \cdots, x^n) \in \mathbb{L}^{n+1} \mid -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = \frac{1}{k},\ x^0 > 0 \}$$

$$\langle x, x \rangle_L = \frac{1}{k}, \quad \boxed{T_x H^n(k) = x^\perp}$$

$$\langle x, x \rangle_L = \frac{1}{k} < 0$$

$$\Rightarrow \quad \langle\ ,\ \rangle_L \Big|_{T_x H^n(k)} : \text{positive definite}$$

$\langle\ ,\ \rangle_L$: induces a Riemannian metric of $H^n(k)$

the Hyperbolic space of constant curvature $k < 0$

Exercise 2-1

Problem (Ex. 2-1)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$, *and set*

$$\boldsymbol{f}\colon D \ni (u,v) \mapsto \boxed{\frac{1}{1-u^2-v^2}(1+u^2+v^2, 2u, 2v)} \in \mathbb{L}^3.$$

- *For each* $(u,v) \in D$,
- *Show that* $\boldsymbol{f}$ *is a bijection from* D *to* $H^3(-1)$.

②

- *Compute* $\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle$, $\langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle$ *and* $\langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle$.
- *For each* $(u,v) \in D$, *find an orthonormal basis* $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *of* $\underbrace{T_{\boldsymbol{x}}H^3(-1)}_{\mathrm{span}(f_u,\, f_v)}$, *where* $\boldsymbol{x} = \boldsymbol{f}(u,v)$.

Exercise 2-2

Problem (Ex. 2-2)

Fix an $(n+1)\times(n+1)$*-orthogonal matrix* A *and set*

$$\varphi\colon S^n(k) \ni \boldsymbol{x} \mapsto A\boldsymbol{x} \in \mathbb{R}^{n+1},$$

where k *is a positive number. Fix* $\boldsymbol{x} \in S^n(k)$ *and take a smooth curve* $\gamma(t)$ *on* $S^n(k)$ *such that* $\gamma(0) = \boldsymbol{x}$ *and set* $\boldsymbol{v} := \dot\gamma(0) \in T_{\boldsymbol{x}} S^n(k)$.

- *Show that* φ *induces a bijection from* $S^n(k)$ *into* $S^n(k)$.
- *Show that* $\boxed{\varphi_*\boldsymbol{v} := \left.\frac{d}{dt}\right|_{t=0} \varphi\circ\gamma = A\boldsymbol{v}}$.
- *Verify that* $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \langle \varphi_*\boldsymbol{v}, \varphi_*\boldsymbol{v}\rangle$.