Advanced Topics in Geometry E1 (MTH.B505)

Riemannian manifolds

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Manifolds

A smooth manifold M is

- a topological space
- a family of local coordinate systems (charts)
- smooth coordinate change

Example (The n-dimensional affine space)

$\mathbb{R}^n := \{(x^1, \dots, x^n)^T ; x^j \in \mathbb{R}, j = 1, 2, \dots, n\}$

Manifolds—submanifolds of $\mathbb{R}^n$

Fact (Implicit Function Theorem)

- $\boldsymbol{F}\colon \mathbb{R}^{n+r} \to \mathbb{R}^r$: *smooth*
- $M := \boldsymbol{F}^{-1}(\mathbf{0}) = \{p \in \mathbb{R}^{n+r}\,;\, \boldsymbol{F}(p) = \mathbf{0}\} \neq \emptyset$
- $\operatorname{rank} d\boldsymbol{F} = r$ *on* M

$\Rightarrow$ M *is an* n*-dimensional submanifold of* $\mathbb{R}^{n+r}$

Example: spheres

- $n \geqq 1$
- $k > 0$
- $F(\boldsymbol{x}) := F(x^0, \ldots, x^n) = \left(\sum j = 0^n (x^j)^2\right) - \frac{1}{k} = \langle \boldsymbol{x}, \boldsymbol{x} \rangle - \frac{1}{k}.$

$$S^n(k) = F^{-1}(\{0\}) \subset \mathbb{R}^{n+1}$$

Smooth functions

$\mathcal{F}(M)$: an algebra consists of smooth functions on M.

- $\mathcal{F}(\mathbb{R}^n)$ is the set of smooth functions of n-variables.
- For an n-dimensional submanifold $M \in \mathbb{R}^N$, an element of $\mathcal{F}(M)$ is a function $f\colon M \to \mathbb{R}$ such that $f \circ \varphi$ is of class C^∞ for any smooth function $\varphi\colon \mathbb{R}^n \to M$.

Tangent spaces

T_pM: the tangent space of a manifold M at $p \in M$.

- the set of "velocity vectors" of curves on M passing through p.
- the set of "directional derivatives" at p.

Example

$T_{\boldsymbol{x}}\mathbb{R}^n = \mathbb{R}^n$.

The tangent space of the sphere

Example

$T_{\boldsymbol{x}} S^n(k) = \boldsymbol{x}^{\perp} = \{\boldsymbol{v} \in \mathbb{R}^n \,;\, \langle \boldsymbol{v}, \boldsymbol{x} \rangle = 0\} \in \mathbb{R}^{n+1}$

Tangent bundles

$TM = \cup_{p \in M} T_p M$; $\pi \colon TM \to M$: the projection

- "vector bundle of rank $n := \dim M$" over M.
- a $2n$-dimensional manifold.

Example

$T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.

The tangent bundle of the sphere

Example

$TS^n(k) = \boldsymbol{x}^\perp = \{(\boldsymbol{x}, \boldsymbol{v}) \in S^n(k) \times \mathbb{R}^n \,;\, \langle \boldsymbol{v}, \boldsymbol{x} \rangle = 0\} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

Vector fields

A vector field X of a manifold M is
a smooth map $X\colon M \to TM$ such that $\pi \circ X = \mathrm{id}$.
$\mathfrak{X}(M) =$ the set of vector fields on M

- $\mathfrak{X}(M)$ is a vector space.
- $\mathfrak{X}(M)$ is an $\mathcal{F}(M)$-module

Vector fields on $\mathbb{R}^n$ and $S^n(k)$

- $\mathfrak{X}(\mathbb{R}^n) = \{X : \mathbb{R}^n \to \mathbb{R}^n; C^\infty\}$
- $\mathfrak{X}(S^n(k)) = \{X = (\boldsymbol{x}, \boldsymbol{v}) : S^n(k) \ni \boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^{2n}; \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0\}$

Riemannian manifolds

Definition

A Riemannian metric g on an n-manifold M is a correspondence $p \mapsto g_p$ of p to an inner product g_p of T_pM, which satisfies the smoothness condition, that is,

$$g(X, Y) : M \ni p \mapsto g_p(X_p, y_p) \in \mathbb{R}$$

is a smooth function for each pair of sooth vector fields (X, Y).

Example

$\mathbb{E}^n := (\mathbb{R}^n, \langle\ ,\ \rangle)$ is a Riemannian manifold, called the Euclidean n-space.

Riemannian submanifold

$M \subset \mathbb{R}^{n+r}$: a submanifold.

- $T_pM \subset T_p\mathbb{R}^{n+r}$.
- The restriction of $\langle\ ,\ \rangle$ to $T_pM \times T_pM$ is an inner product.

$\Rightarrow \langle\ ,\ \rangle$ induces a Riemannian metric on M, called the induced metric.

Example

$S^n(k) \subset \mathbb{R}^{n+1}$ is a Riemannian manifold with the induced metric from $\mathbb{R}^{n+1}$, called the sphere of constant curvature k.

The hyperbolic space

Recall: the Lorentz-Minkowski vector space $\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$.

- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_L := -x^0 y^0 + \sum_{j=1}^{n} x^j y^j$:

  $(\boldsymbol{x} = (x^0, x^1, \dots, x^n)^T,\ \boldsymbol{y} = (y^0, y^1, \dots, y^n)^T)$.

  The canonical Lorentzian "inner product".

For $k < 0$,

$$H^n(k) := \{\boldsymbol{x} = (x^0, \dots,^{n_1})^T\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = 1/k, x^0 > 0\}$$

- $T_{\boldsymbol{x}} H^n(k) = \{\boldsymbol{v} \in \mathbb{L}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{v} \rangle\, 0\}$

Fact

The restriction of $\langle\ ,\ \rangle_L$ to $T_{\boldsymbol{x}} H^n(k) \times T_{\boldsymbol{x}} H^n(k)$ is a positive definite inner product, which induces the Riemannian metric $\langle\ ,\ \rangle_L$ to $H^n(k)$. The Riemannian manifold $(H^n(k), \langle\ ,\ \rangle_L)$ is called the hyperbolic n-space of constant curvature k.

Exercise 2-1

Problem (Ex. 2-1)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$, *and set*

$$\boldsymbol{f}: D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3.$$

- *For each* $(u,v) \in D$,
- *Show that* $\boldsymbol{f}$ *is a bijection from* D *to* $H^3(-1)$.
- *Compute* $\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle$, $\langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle$ *and* $\langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle$.
- *For each* $(u,v) \in D$, *find an orthonormal basis* $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *of* $T_{\boldsymbol{x}}H^3(-1)$, *where* $\boldsymbol{x} = \boldsymbol{f}(u,v)$.

Exercise 2-2

Problem (Ex. 2-2)

Fix an $(n+1) \times (n+1)$*-orthogonal matrix* A *and set*

$$\boldsymbol{\varphi}\colon S^n(k) \ni \boldsymbol{x} \mapsto A\boldsymbol{x} \in \mathbb{R}^{n+1},$$

where k *is a positive number. Fix* $\boldsymbol{x} \in S^n(k)$ *and take a smooth curve* $\gamma(t)$ *on* $S^n(k)$ *such that* $\gamma(0) = \boldsymbol{x}$ *and set* $\boldsymbol{v} := \dot{\gamma}(0) \in T_{\boldsymbol{x}}S^n(k)$*.*

- *Show that* $\boldsymbol{\varphi}$ *induces a bijection from* $S^n(k)$ *into* $S^n(k)$*.*
- *Show that* $\boldsymbol{\varphi}_*\boldsymbol{v} := \left.\frac{d}{dt}\right|_{t=0} \boldsymbol{\varphi} \circ \gamma = A\boldsymbol{v}$*.*
- *Verify that* $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = \langle \boldsymbol{\varphi}_*\boldsymbol{v}, \boldsymbol{\varphi}_*\boldsymbol{v} \rangle$*.*