Advanced Topics in Geometry E1 (MTH.B505)

Pseudo Riemannian manifolds

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Exercise 2-1

Parametrization of the hyperbolic plane

Problem (Ex. 2-1)

(can be generalized to higher dim case)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$, *and set*

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3.$$

For each $(u,v) \in D$,

- *Show that* $\boldsymbol{f}$ *is a bijection from* D *to* $H^3(-1)$.
- *Compute* $\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle$, $\langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle$ *and* $\langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle$.
- *For each* $(u,v) \in D$, *find an orthonormal basis* $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *of* $T_{\boldsymbol{x}}H^3(-1)$, *where* $\boldsymbol{x} = \boldsymbol{f}(u,v)$.

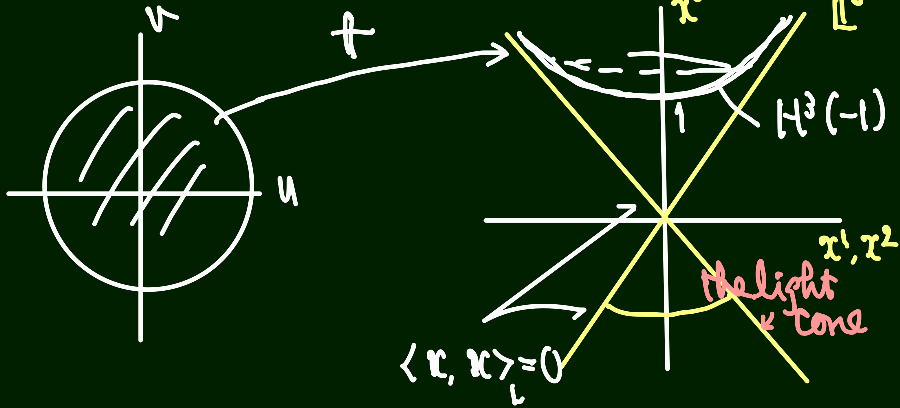
Exercise 2-1

$$D = \{(u,v) \mid u^2+v^2<1\} \subset \mathbb{R}^2$$

$$\mathbb{L}^3 = (\mathbb{R}^3, \langle\, ,\, \rangle_L) \qquad -(x^0)^2+(x^1)^2+(x^2)^2$$

$$f : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right) \in \mathbb{L}^3. \qquad = -1$$

$$H^2(-1) = \{ x = (x^0, x^1, x^2) \in \mathbb{L}^3 \,;\, \langle x, x\rangle_L = -1,\ x^0>0 \}$$



Exercise 2-1

analogue of the stereographic proj

$H^3(-1)$

$$f : D \ni (u, v) \mapsto \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right) \in \cancel{\mathbb{L}}^3.$$

$= (x^0, x^1, x^2)$

$\because \left(x^0 > 0,\ -(x^0)^2 + (x^1)^2 + (x^2)^2 = -1\right)$

f is a bijection (i.e. 1 to 1, onto)

$\because\ H^3(-1) \ni (x^0, x^1, x^2) \longmapsto \frac{1}{1+x^0}(x^1, x^2) \in D$

inverse

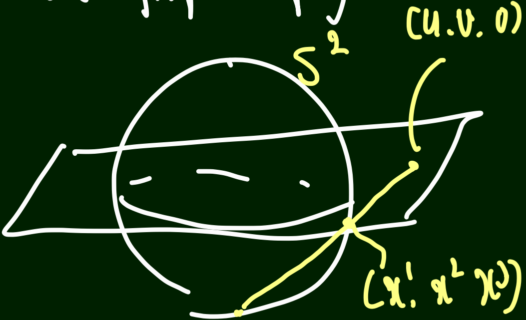
exercise

Exercise 2-1

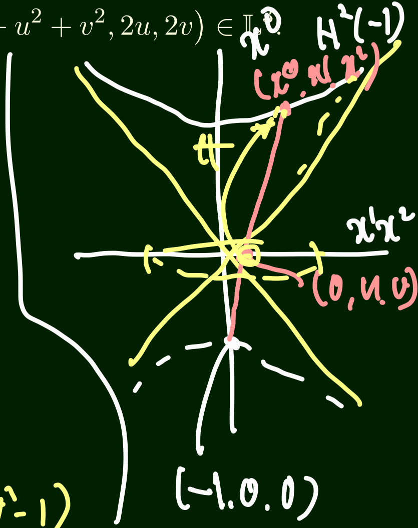
Poincaré disc model

$$f : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right) \in \mathbb{L}^3$$

Stereographic proj



$$x = \frac{1}{1+u^2+v^2}\left(2u, 2v, u^2+v^2-1\right)$$



$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right) \in \mathbb{L}^3.$$

$$\begin{cases} f_u = \dfrac{2}{(1-u^2-v^2)^2}\left(2u,\ 1+u^2-v^2,\ 2uv\right) \\ f_v = \dfrac{2}{(1-u^2-v^2)^2}\left(2v,\ 2uv,\ 1-u^2+v^2\right) \end{cases}$$

$$\langle f_u, f_u \rangle_L = \langle f_v, f_v \rangle_L = \frac{4}{(1-u^2-v^2)^2} > 0$$

$$\langle f_u, f_v \rangle_L = 0 \qquad \frac{4}{(1-u^2-v^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \text{positive definite}$$

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right) \in \mathbb{L}^3.$$

For each (u,v), $\underbrace{\operatorname{Span}\{f_u, f_v\}}_{T_x H^2(-1)}$ $=: W$ is "positive definite"

$x = f(u,v)$

$\left(\langle\, , \rangle_L \big|_{W \times W} \text{ is positive.} \right)$

✓ $e_1 = \dfrac{1-u^2-v^2}{2} f_u$, $e_2 = \dfrac{1-u^2-v^2}{2} f_v$

$\Rightarrow [e_1, e_2] = \mathrm{ON}$

Exercise 2-2

Isometry of $S^n(k)$

Problem (Ex. 2-2)

Fix an $(n+1) \times (n+1)$*-orthogonal matrix* A *and set*

$$\boldsymbol{\varphi}\colon S^n(k) \ni \boldsymbol{x} \mapsto A\boldsymbol{x} \in \mathbb{R}^{n+1},$$

where k *is a positive number. Fix* $\boldsymbol{x} \in S^n(k)$ *and take a smooth curve* $\gamma(t)$ *on* $S^n(k)$ *such that* $\gamma(0) = \boldsymbol{x}$ *and set* $\boldsymbol{v} := \dot{\gamma}(0) \in T_{\boldsymbol{x}} S^n(k)$.

- *Show that* $\boldsymbol{\varphi}$ *induces a bijection from* $S^n(k)$ *into* $S^n(k)$.
- *Show that* $\boldsymbol{\varphi}_* \boldsymbol{v} := \left.\frac{d}{dt}\right|_{t=0} \boldsymbol{\varphi} \circ \gamma = A\boldsymbol{v}$.
- *Verify that* $\boxed{\langle \boldsymbol{v}, \boldsymbol{v} \rangle = \langle \boldsymbol{\varphi}_* \boldsymbol{v}, \boldsymbol{\varphi}_* \boldsymbol{v} \rangle}$.

$S^n(k) = \{ x \in \mathbb{E}^{n+1} ; \langle x, x \rangle = \frac{1}{k} \} \quad (k > 0)$

A : $(n+1) \times (n+1)$ orthogonal matrix

Recall A : orthogonal $\quad AA^T = A^T A = I$

identity

$\Rightarrow \langle Ax, Ay \rangle = \langle x, y \rangle$

Euclidean inner product $\quad$ for $\forall\, x, y \in \mathbb{E}^{n+1}$

$\Rightarrow \quad \varphi : S^n(k) \ni x \longmapsto \varphi(x) = Ax \in S^n(k)$

$\varphi^{-1}(y) = A^{-1} y \; (= A^T y)$

$\varphi : S^n(k) \longrightarrow S^n(k)$: a bijection

☆ $x \in S^n(k)$, $v \in T_x S^n(k)$

$$v = \frac{d}{dt}\Big|_{t=0} {}^{\exists}\gamma(t)$$

- γ : a curve in $S^n(k)$
- $\gamma(0) = x$

☆ $\hat{\gamma}(t) = \varphi(\gamma(t))$ $\quad \hat{\gamma}(0) = \varphi(x)$

$\hat{\gamma}$: a curve in $S^n(k)$

$$\frac{d}{dt}\Big|_{t=0} \hat{\gamma}(t) =: \underline{\underline{\varphi_* v}}$$

$\left(\varphi_* : T_x M \to T_{\varphi(x)} M\right)$

$$\frac{d}{dt}\Big|_0 A\gamma(t) = A\dot{\gamma}(0) = \underline{\underline{Av}}$$

$\varphi_* v = A v$

$\Rightarrow \underline{\langle \varphi_* v, \varphi_* w \rangle} = \langle Av, Aw \rangle$

$= \underline{\langle v, w \rangle}$

af

φ is an isometry.

Definition

(bijective)

Let (M, g) be a Riemannian manifold. A smooth map $\varphi\colon M \to M$ is an isometry if

$$g(X, Y) = g(\varphi_* X, \varphi_* Y)$$

"preserve the inner product"

holds for all tangent vectors X and Y of M.

Fact

Set of the isometries, called the isometry group, is a group with respect to composition of maps.

Fact

The isometry group of $S^n(k)$ contains the orthogonal group $\mathrm{O}(n+1)$.

$O(n+1) = \{ (n+1)\times(n+1)\text{-orthogonal matrix} \}$

the orthogonal group

Isometry group of $S^n(k) \supset O(n+1)$

$\parallel$ Ex 2-2

$\boxed{\text{Fact}}$ $\mathrm{Isom}\, S^n(k) = O(n+1)$

(we prove later)