Advanced Topics in Geometry E1 (MTH.B505)

Pseudo Riemannian manifolds

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Exercise 2-1

Problem (Ex. 2-1)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$, *and set*

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3.$$

For each $(u,v) \in D$,

- *Show that* $\boldsymbol{f}$ *is a bijection from* D *to* $H^3(-1)$.
- *Compute* $\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle$, $\langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle$ *and* $\langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle$.
- *For each* $(u,v) \in D$, *find an orthonormal basis* $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *of* $T_{\boldsymbol{x}}H^3(-1)$, *where* $\boldsymbol{x} = \boldsymbol{f}(u,v)$.

Exercise 2-1

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3.$$

Exercise 2-2

Problem (Ex. 2-2)

Fix an $(n+1)\times(n+1)$*-orthogonal matrix* A *and set*

$$\boldsymbol{\varphi}\colon S^n(k) \ni \boldsymbol{x} \mapsto A\boldsymbol{x} \in \mathbb{R}^{n+1},$$

where k *is a positive number. Fix* $\boldsymbol{x} \in S^n(k)$ *and take a smooth curve* $\gamma(t)$ *on* $S^n(k)$ *such that* $\gamma(0) = \boldsymbol{x}$ *and set* $\boldsymbol{v} := \dot{\gamma}(0) \in T_{\boldsymbol{x}}S^n(k)$*.*

- *Show that* $\boldsymbol{\varphi}$ *induces a bijection from* $S^n(k)$ *into* $S^n(k)$*.*
- *Show that* $\boldsymbol{\varphi}_*\boldsymbol{v} := \frac{d}{dt}\Big|_{t=0} \boldsymbol{\varphi}\circ\gamma = A\boldsymbol{v}$*.*
- *Verify that* $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \langle \boldsymbol{\varphi}_*\boldsymbol{v}, \boldsymbol{\varphi}_*\boldsymbol{v}\rangle$*.*

Definition

Let (M, g) be a Riemannian manifold. A smooth map $\varphi\colon M \to M$ is an isometry if

$$g(X, Y) = g(\varphi_* X, \varphi_* Y)$$

holds for all tangent vectors X and Y of M.

Fact

Set of the isometries, called the isometry group, is a group with respect to composition of maps.

Fact

The isometry group of $S^n(k)$ contains the orthogonal group $\mathrm{O}(n+1)$.