• $H^n(k) \subset \mathbb{L}^{n+1}$: the Lorentz Minkowski space

Advanced Topics in Geometry E1 (MTH.B505)

Pseudo Riemannian manifolds

manifold
"generalized" inner product

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2023/05/02

Non-degenerate bilinear form

V: n-dimensional vector space over $\mathbb{R}$.
$\langle\ ,\ \rangle$: a symmetric bilinear form

Def of Lecture 1
inner product
$\Leftrightarrow$ positive definite
i.e. $\langle x, x \rangle > 0$ if $x \neq 0$

Definition

$\langle\ ,\ \rangle$ is $\boxed{\text{non-degenerate}}$ iff

$$\underline{\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0} \quad \underline{\text{for all } \boldsymbol{y} \in V} \quad \Leftrightarrow \quad \boldsymbol{x} = \boldsymbol{0}.$$

$\Rightarrow$

Fact

A positive definite symmetric bilinear form is non-degenerate.

$\langle x, y \rangle = 0$ for $\forall y \Rightarrow \langle x, x \rangle = 0$

$\Rightarrow x = 0$

positive definite

Inner product

An $\boxed{\text{inner product}}$ (in an extended sense) =
a non-degenerate symmetric bilinear form

Example $(m \geq 0,\ r \geq 0\ \ m + r = n > 0)$

$$x = (x^1, \cdots x^n)^T \in \mathbb{R}^n$$

$$y = (y^1 \cdots y^n)^T \in \mathbb{R}^n$$

$$\langle x, y \rangle = x^T \begin{bmatrix} \overbrace{-1 \ \ddots \ -1}^{r} & 0 \\ 0 & \overbrace{1 \ \ddots \ 1}^{m} \end{bmatrix} y \qquad (*)$$

$\langle \, , \, \rangle$ a symm. bilinear form

• $\langle \, , \, \rangle$: non-degenerate

$\langle x, y \rangle = 0$ for $\forall y$

$$\Rightarrow \underbrace{(x^1, \cdots, x^n) \begin{pmatrix} -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}} \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = 0 \quad \forall y$$

$$\Rightarrow \underbrace{(-x^1, \cdots, -x^r, x^{r+1}, \cdots x^n)}_{\tilde{x}^T} \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = 0 \quad \forall y$$

$$\Rightarrow \langle \tilde{x}, y \rangle_{\text{Euclidean}} = 0 \text{ for } \forall y$$

$$\Rightarrow \tilde{x} = 0 \qquad \Rightarrow x = 0 \quad \square$$

Pseudo Euclidean spaces

Example

$m \geqq 1$; $r \geqq 0$; $m+r>0$

$\mathbb{E}^{m+r}_r := (\mathbb{R}^{m+r}, \langle\ ,\ \rangle)$, where $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -\left(\sum_{j=1}^{r} x^j y^j\right) + \left(\sum_{l=r+1}^{m+r}\right.$

$\Rightarrow \langle\ ,\ \rangle$: a non-degenerate symmetric bilinear form

$x^T \begin{bmatrix} -1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} y$

1. $r = 0$: the Euclidean space (positive definite)
2. $r = 1$: the Lorentz-Minkowski space (indefinite) $\mathbb{L}^n$
3. $r = n = \dim V$: negative definite

Restriction of the inner product

$(V, \langle , \rangle)$

$W \subset$ non-deg

$W \subset V$: a subspae
$\langle\ ,\ \rangle|_{W\times W}$: the restriction of the inner product $\langle\ ,\ \rangle$. a symmetric bilinear form on W.

Lemma

If $\langle\ ,\ \rangle$ is positive (negative) definite, so is its restriction $\langle\ ,\ \rangle|_{W\times W}$.

· $\underline{\langle x, x\rangle > 0}$ for $\forall x \in W \setminus \{0\}$

"neiter pos definite nor neg. definite"

= "indefinite"

Example $\mathbb{L}^3$

$\langle\, ,\, \rangle$

$W \subset \mathbb{R}^3 = \mathbb{E}^3_1$: a linear subspace given by

$\varphi\begin{pmatrix}1\\0\\0\end{pmatrix} = -1 \neq 0$

$$W = x^\perp = \{v\,;\,\langle x, v\rangle = 0\} \qquad (x := (1,1,0)^T)$$

$\dim W = 2$

$\varphi : \mathbb{E}^3_1 \ni v \longmapsto \varphi(v) = \langle x, v\rangle \in \mathbb{R}$: linear

$x := (1,1,0)^T \in W$.

$\boxed{\langle x, x\rangle = 0}$

- $\langle\, ,\, \rangle|_{W\times W}$: $\boxed{\text{degenerates.}}$

$W = \operatorname{Ker}\varphi \subset \mathbb{E}^3_1$ linear subsp.

$\dim W = \dim \mathbb{E}^3_1 - \boxed{\dim \operatorname{Im}\varphi = 1}$

Signature

$\langle\ ,\ \rangle$: an inner product on V $(\dim V = n)$

$m = \max\{\dim W\ ;\ W \subset V$: a subspace, $\langle\ ,\ \rangle|_{W\times W}$ is positive definite$\}$
$r = \max\{\dim W\ ;\ W \subset V$: a subspace, $\langle\ ,\ \rangle|_{W\times W}$ is negative definite$\}$

Lemma

$m + r = n.$

(m, r): the signature of $(V, \langle\ ,\ \rangle)$

$\mathbb{E}^{m+r}_{r}$: signature (m, r)

Signature

$(V, \langle\ ,\ \rangle)$: of signature (m, r); $m + r = n = \dim V$.

- $r = 0$: a positive definite inner product
- $r = 1$: a Lorentzian inner product

$\mathbb{E}_r^{m+r}$ $(m.r)$

$\exists$ one dim subspace on which $\langle , \rangle$ is neg. definite

$\nexists$ two " on which $\langle , \rangle$ is neg. def

Orthogonal Complement

$x \in \mathbb{L}^{n+1} = \mathbb{E}^{n+1}_1$ $\quad$ $m = n,\ r = 1$

$\boxed{\langle x, x \rangle = -1} \Rightarrow x^\perp$: pos-def (cf. Ex 1-2)

$\langle\ ,\ \rangle$: an inner product of signature (m, r) on V $(m + r = n)$

Proposition

$\boldsymbol{x} \in V$: $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$.

(m, r)

- *$W := \boldsymbol{x}^\perp$ is an $(n-1)$-dimensional subspace of V*
- *$\langle\ ,\ \rangle|_{W \times W}$ is of signature $(m, r-1)$.*

non-degenerate

Pseudo Riemannian manifolds

Pseudo Riem. metric
= ~~pos def~~ indefinite inner product
on tang sp.

Definition

A pseudo Riemannian metric g of signature (m, r) on a connected n $(= m + r)$-manifold M is a correspondence $p \mapsto g_p$ of p to an inner product g_p of signature (m, r) on T_pM, which satisfies the smoothness condition, that is,

$$g(X, Y) : M \ni p \mapsto g_p(X_p, Y_p) \in \mathbb{R}$$

is a smooth function for each pair of sooth vector fields (X, Y). A connected n-manifold M endowed with a pseudo Riemannian metric g is called a pseudo Riemannian manifold

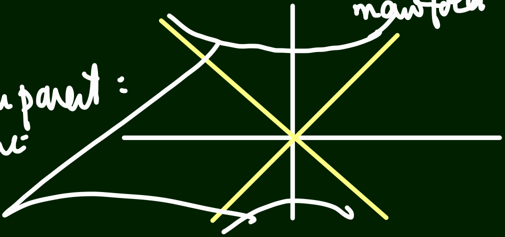
$\mathbb{E}^{n,1}$

$$M^n(a) := \{x \in \mathbb{E}^{n+1}_1 ; \langle x, x \rangle = a\}.$$

- $F(x) := \langle x, x \rangle - a$ $\qquad M^n(a) = F^{-1}(\{0\})$
- $dF = 0 \Leftrightarrow x = 0$
- If $a \neq 0$ $\qquad dF \neq 0$ on $M^n(a)$: $M^n(a)$ smooth manifold

x^0

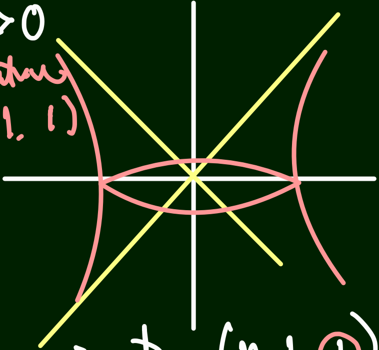
$\boxed{a<0}$: each connected component : hyperbolic space $M^n(a)$ $(a<0)$



$a>0$ $\quad \langle x, x \rangle = a > 0$

$T_x M^n(a) = x^{\perp}$ signature $(n-1, 1)$

$\subset \mathbb{L}^{n+1}$ $(n, 1)$



$\langle , \rangle$ on $T_x M^n(a)$ has signature $(n-1, 1)$

$(M^n(a), \langle , \rangle)$: a Lorentzian manifold

n-dim <u>de Sitter space</u> of curvature $\frac{1}{a}$

Example

$(n-1, 2)$

$$M^n(a) := \{x \in \mathbb{E}_2^{n+1}\,;\, \langle x, x \rangle = a\}.$$

$a < 0$

$T_x M^n(a) \cong x^\perp$

$\langle\, , \rangle$: of signature $(n-1, 1)$

$(M^n(a), \langle\, , \rangle)$: a Lorentzian mfd

(n-dimensional anti de Sitter space of curvature $\frac{1}{a}$)

Exercise 3-1

Problem (Ex. 3-1)

Let $\mathrm{O}(2,1)$ *be the set of* 3×3*-matrices satisfying*

Lorentzian group

$$\mathrm{O}(2,1) := \left\{ A = (a_{ij})_{i,j=0}^{2}\, ; \, A^T Y A = Y \right\} \quad \left(Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

$a_{00} \geqq 1 \quad a_{00} \leqq -1$

$x \mapsto Ax$

$\langle x, y \rangle = x^T Y y$

- *Show that* $|\det A| = 1$ *for* $A \in \mathrm{O}(2,1)$.
- *Show that* $|a_{00}| \geqq 1$ *for* $A = (a_{ij})$.
- *Show that the liner transformation induced by* $A \in \mathrm{O}(2,1)$ *preserves the inner product* $\langle\ ,\ \rangle$ *of* $\mathbb{E}^3_1$.
- $\mathrm{SO}_+(2,1) := \{ A = (a_{ij}) \in \mathrm{O}(2,1)\, ; \, \det A = 1, a_{00} \geqq 1 \}$ *induces a bijection from the hyperbolic space* $H^2(k) \subset \mathbb{E}^3_1$ *to itself, where* $k < 0$.

Exercise 3-2

Problem (Ex. 3-2)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$*, and set*

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3 = \mathbb{E}^3_1,$$

and take an orthonormal basis $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *of* $T_{\boldsymbol{x}}H^3(-1)$*, where* $\boldsymbol{x} = \boldsymbol{f}(u,v)$*.*

ON

- *Verify that, for each* $(u,v) \in D$*,* $\underbrace{[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]}$ *is a basis of* $\mathbb{R}^3$*, where* $\boldsymbol{e}_0 = \boldsymbol{f}$*.*
- *Express the derivatives* $(\boldsymbol{e}_j)_u$ *and* $(\boldsymbol{e}_j)_v$ $(j = 0,1,2)$ *as linear combinations of* $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$*.*

$(\boldsymbol{e}_j)_u = \square\, \boldsymbol{e}_0 + \square\, \boldsymbol{e}_1 + \square\, \boldsymbol{e}_2$