Advanced Topics in Geometry E1 (MTH.B505)

Pseudo Riemannian manifolds

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Non-degenerate bilinear form

V: n-dimensional vector space over $\mathbb{R}$.

$\langle\ ,\ \rangle$: a symmetric bilinear form

Definition

$\langle\ ,\ \rangle$ is non-degenerate iff

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0 \quad \text{for all } \boldsymbol{y} \in V \quad \Leftrightarrow \quad \boldsymbol{x} = \boldsymbol{0}.$$

Fact

A positive definite symmetric bilinear form is non-degenerate.

Inner product

An inner product (in an extended sense) $=$

a non-degenerate symmetric bilinear form

Pseudo Euclidean spaces

Example

$n \geqq 1$; $r \geqq 0$;

$$\mathbb{E}^{m+r}_r := (\mathbb{R}^{m+r}, \langle\ ,\ \rangle), \qquad \text{where} \qquad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\left(\sum_{j=1}^{r} x^j y^j\right) + \left(\sum_{l=r+1}^{m+r} x^l y^l\right.$$

$\Rightarrow \langle\ ,\ \rangle$: a non-degenerate symmetric bilinear form

1. $r = 0$: the Euclidean space (positive definite)
2. $r = 1$: the Lorentz-Minkowski space (indefinite)
3. $r = n = \dim V$: negative definite

Restriction of the inner product

$W \subset V$: a subspae

$\langle\ ,\ \rangle|_{W\times W}$: the restriction of the inner product $\langle\ ,\ \rangle$. a symmetric bilinear form on W.

Lemma

If $\langle\ ,\ \rangle$ is positive (negative) definite, so is its restriction $\langle\ ,\ \rangle|_{W\times W}$.

Example

$W \subset \mathbb{R}^3 = \mathbb{E}^3_1$: a linear subspace given by

$$W = \boldsymbol{x}^\perp = \{\boldsymbol{v}\,;\,\langle \boldsymbol{x}, \boldsymbol{v}\rangle = 0\} \qquad (\boldsymbol{x} := (1,1,0)^T)$$

$\dim W = 2$

$$\boldsymbol{x} := (1,1,0)^T \in W.$$

$\langle\ ,\ \rangle_{W\times w}$: degenerates.

Signature

$\langle\ ,\ \rangle$: an inner product on V $(\dim V = n)$

$$m := \max\{\dim W\,;\, W \subset V : \text{ a subspace}, \langle\ ,\ \rangle|_{W\times W} \text{ is positive definite}\},$$
$$r := \max\{\dim W\,;\, W \subset V : \text{ a subspace}, \langle\ ,\ \rangle|_{W\times W} \text{ is negative definite}\}.$$

Lemma

$m + r = n.$

(m, r): the signature of $(V, \langle\ ,\ \rangle)$

Signature

$(V, \langle\ ,\ \rangle)$: of signature (m, r); $m + r = n = \dim V$.

- $r = 0$: a positive definite inner product
- $r = 1$: a Lorentzian inner product

$\mathbb{E}_r^{m+r}$

Orthogonal Complement

$\langle\ ,\ \rangle$: an inner product of signature (m, r) on V $(m + r = n)$

Proposition

$\boldsymbol{x} \in V$: $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$.

- $W := \boldsymbol{x}^{\perp}$ *is an* $(n-1)$*-dimensional subspace of* V
- $\langle\ ,\ \rangle|_{W \times W}$ *is of signature* $(m, r-1)$.

Pseudo Riemannian manifolds

Definition

A pseudo Riemannian metric g of signature (m, r) on a connected n $(= m + r)$-manifold M is a correspondence $p \mapsto g_p$ of p to an inner product g_p of signature (m, r) on T_pM, which satisfies the smoothness condition, that is,

$$g(X, Y) : M \ni p \mapsto g_p(X_p, Y_p) \in \mathbb{R}$$

is a smooth function for each pair of sooth vector fields (X, Y).
A connected n-manifold M endowed with a pseudo Riemannian metric g is called a pseudo Riemannian manifold

Example

$$M^n(a) := \{\boldsymbol{x} \in \mathbb{E}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = a\}.$$

Example

$$M^n(a) := \{\boldsymbol{x} \in \mathbb{E}^{n+1}_2\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = a\}.$$

Exercise 3-1

Problem (Ex. 3-1)

Let $\mathrm{O}(2,1)$ *be the set of* 3×3*-matrices satisfying*

$$\mathrm{O}(2,1) := \left\{A = (a_{ij})_{i,j=0}^{2}\,;\, A^T Y A = Y\right\} \quad \left(Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right).$$

- *Show that* $|\det A| = 1$ *for* $A \in \mathrm{O}(2,1)$.
- *Show that* $|a_{00}| \geqq 1$ *for* $A = (a_{ij})$.
- *Show that the liner transformation induced by* $A \in \mathrm{O}(2,1)$ *preserves the inner product* $\langle\ ,\ \rangle$ *of* $\mathbb{E}^3_1$.
- $\mathrm{SO}_+(2,1) := \{A = (a_{ij}) \in \mathrm{O}(2,1)\,;\, \det A = 1, a_{00} \geqq 1\}$ *induces a bijection from the hyperbolic space* $H^2(k) \subset \mathbb{E}^3_1$ *to itself, where* $k < 0$.

Exercise 3-2

Problem (Ex. 3-2)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$, *and set*

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3 = \mathbb{E}^3_1,$$

and take an orthonormal basis $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *of* $T_{\boldsymbol{x}}H^3(-1)$, *where* $\boldsymbol{x} = \boldsymbol{f}(u,v)$.

- *Verify that, for each* $(u,v) \in D$, $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$ *is a basis of* $\mathbb{R}^3$, *where* $\boldsymbol{e}_0 = \boldsymbol{f}$.
- *Express the derivatives* $(\boldsymbol{e}_j)_u$ *and* $(\boldsymbol{e}_j)_v$ *(*$j = 0,1,2$*) as linear combinations of* $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$.