Advanced Topics in Geometry E1 (MTH.B505)

Riemannian connection for submanifolds

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Problem (Ex. 3-1)

⇒ O(3) (orthogonal group)

Let $\boxed{\mathrm{O}(2,1)}$ *be the set of* 3×3*-matrices satisfying* id

3×3 matrices

$$\mathrm{O}(2,1) := \left\{ A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \in \mathrm{M}_3(\mathbb{R})\,;\, A^T Y A = Y \left(Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right\}.$$

a group

w.r.to matrix multiplication

"pseudo orthogonal group" of sign (2.1)

We know:

n

identity

- $O(3) = \{ A \in M_3(\mathbb{R});\ A^T A = I \}$

$\Rightarrow$
- $O(3)$ is a group
- $O(3) \subset M_3(\mathbb{R}) \cong \mathbb{R}^9$: 3 dim submanifold
  compact
  disconnected
- $A \in O(3)$

connected components

$\Rightarrow \det A = \pm 1$

$\det I = 1, \quad \det(-I) = -1$

- $SO(3) = \{ A \in O(3);\ \det A > 0 \} \neq \emptyset$

$\}$ $\cdots$ $\det A < 0 \} \neq \emptyset$

$\Rightarrow \det O(3) = \{ \pm 1 \}$: disconnected

Exercise 3-1

a_{00} \ det	+	−
+	$SO_+(2,1)$	
−		

$\left(A^T Y A = Y, \ \det Y = -1 \right)$

Problem (Ex. 3-1)

- *Show that* $|\det A| = 1$ *for* $A \in \mathrm{O}(2,1)$.
- *Show that* $|a_{00}| \geqq 1$ *for* $A = (a_{ij})$.
- *Show that the liner transformation induced by* $A \in \mathrm{O}(2,1)$ *preserves the inner product* $\langle\ ,\ \rangle$ *of* $\mathbb{E}^3_1$.
- $\mathrm{SO}_+(2,1) := \{A = (a_{ij}) \in \mathrm{O}(2,1)\,;\ \det A = 1, a_{00} \geqq 1\}$ *induces a bijection from the hyperbolic space* $H^2(k) \subset \mathbb{E}^3_1$ *to itself, where* $k < 0$.

O(2,1) consists of 4 connected comp.

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} = (a_0 \ a_1 \ a_2)$$

Lorentzian inner product

$$A \in O(2,1) \Rightarrow \boxed{\begin{aligned} \langle a_0, a_0 \rangle \\ = a_0^T Y a_0 = -1 \end{aligned}} \ \star$$

orthonormal basis of $\langle , \rangle$

$$\langle a_i, a_i \rangle = +1 \quad (i=1,2)$$

$$\langle a_i, a_j \rangle = 0 \quad \text{otherwise}$$

$$\star \quad -a_{00}^2 + a_{10}^2 + a_{20}^2 = -1$$

$$a_{00}^2 = 1 + a_{10}^2 + a_{20}^2 \geq 1 \quad \therefore |a_{00}| \geq 1$$

in $\mathbb{E}^3_1$ $\quad \langle x, y \rangle = x^T Y y \quad Y = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

$$\langle Ax, Ay \rangle = x^T A^T Y A y$$

$$= x^T Y y \qquad A \in O(2,1)$$

$$= \langle x, y \rangle$$

Thm $\varphi : \mathbb{E}^3_1 \longrightarrow \mathbb{E}^3_1$ · preserves the inner prod.

$\Rightarrow \varphi(x) = Ax \quad A \in O(2.1)$

$H^2(k) = \{ x \in E^3_1 \; ; \; \boxed{\langle x, x \rangle = \frac{1}{k}} , \; \boxed{x_0 > 0} \} \quad k < 0$

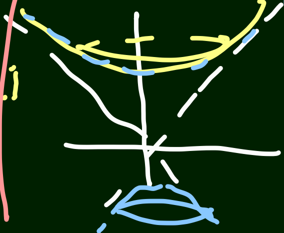
$a_{00} > 0$

x^0

- $H^2(k) \ni x \xrightarrow{\text{bijection}} Ax \in E^3_1$

$H^3(k)$

$A \in SO(2,1)$



preserved by $x \mapsto Ax$

$\boxed{\text{the 1st component of } Ax} = x_0 a_{00} + x_1 a_{01} + x_2 a_{02} \; (> 0)$

$$\left(\begin{array}{l} \bullet \; -a_{00}^2 + a_{01}^2 + a_{02}^2 = -1 \\ \bullet \; -x_0^2 + x_1^2 + x_2^2 = \frac{1}{k} \end{array} \right)$$

$$A^T Y A = Y$$

- $-a_{00}^2 + a_{10}^2 + a_{20}^2 = -1$
- $-a_{00}^2 + a_{01}^2 + a_{02}^2 = -1$

$A \in O(2.1) \Rightarrow A^T \in O(2.1)$

- $A^T Y A Y = I$

$A^T Y = (AY)^{-1}$

$A Y A^T Y = I$

$A Y A^T = Y \quad \therefore A^T \in O(2.1)$

$X = A^{-1}$

$\Longleftrightarrow$ def

$AX = XA = I$

$\Longleftrightarrow$ Thm

$AX = I$

$XA = I$

Exercise 3-1

$$\mathrm{O}(n,1) := \{A \in \mathrm{M}_n(\mathbb{R})\,;\, A^T Y A = Y\}, \qquad Y = \mathrm{diag}(-1, 1, \dots, 1)$$

- $f\colon \mathbb{E}_1^{n+1} \to \mathbb{E}_1^{n+1}$: a bijection preserving the Minkowski inner product
  $\Rightarrow f(\boldsymbol{x}) = A\boldsymbol{x}$ $(A \in \mathrm{O}(n,1))$
- $A \in \mathrm{O}(n,1) \Rightarrow \det A = \pm 1.$
- $A = (a_{ij}) \in \mathrm{O}(n,1) \Rightarrow |a_{00}| \geqq 1$

Exercise 3-2

Problem (Ex. 3-1)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$, *and set*

$H^2(-1)$

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3 = \mathbb{E}^3_1,$$

and take an orthonormal basis $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *of* $T_{\boldsymbol{x}}H^3(-1)$, *where* $\boldsymbol{x} = \boldsymbol{f}(u,v)$.

- *Verify that, for each* $(u,v) \in D$, $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$ *is a basis of* $\mathbb{R}^3$, *where* $\boldsymbol{e}_0 = \boldsymbol{f}$.
- *Express the derivatives* $(\boldsymbol{e}_j)_u$ *and* $(\boldsymbol{e}_j)_v$ $(j = 0,1,2)$ *as linear combinations of* $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$.

$$f : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\big(1+u^2+v^2, 2u, 2v\big) \in \mathbb{L}^3 = \mathbb{E}^3_1,$$

$f(D) = H^2(-1)$

(Poincaré model)

Tangent space of $H^2(-1)$ at $x = f(u,v)$ is spanned by f_u, f_v

$\langle f, f \rangle = -1$

$\Rightarrow$ • $\langle f_u, f_u \rangle = \dfrac{4}{(1-u^2-v^2)^2} = \langle f_v, f_v \rangle$

• $\langle f_u, f_v \rangle =$

• $\langle f_u, f \rangle = \frac{1}{2}\langle f, f \rangle_u = \frac{1}{2}(-1)_u = 0$

$\langle f_v, f \rangle = 0$

$\Rightarrow$ $\big[\, e_0 \;\; e_1 \;\; e_2 \,\big]^{(u,v)}$: orthonormal frame on $\mathbb{E}^3_1$

$e_0 = f$, $\quad e_1 = \dfrac{1-u^2-v^2}{2} f_u$, $\quad e_2 = \dfrac{1-u^2-v^2}{2} f_v$

tangential part

$\langle e_0, e_0 \rangle = -1$

$(e_0)_u = \boxed{0}\, e_0 + \boxed{\odot}\, e_1 + \boxed{\cdot}\, e_2$

$(e_1)_u = \boxed{\circ}\, e_0 + \boxed{0}\, e_1 + \boxed{\cdot}\, e_2$

$\langle e_1, e_1 \rangle = 1$

$(e_2)_u =$

$D_{\partial/\partial u} e_1$

$\nabla_{\partial/\partial u} e_1$

$\odot = \langle (e_0)_u, e_1 \rangle = \langle f_u, \frac{1 - u^2 - v^2}{2} f_{v} \rangle$

$= \cdots$

$\circ = \langle (e_1)_u, e_0 \rangle = \cancel{\langle e_1, e_0 \rangle_u} - \langle e_1, (e_0)_u \rangle$

$= -\odot$

"covariant derivative" / "Riemannian connection"