Advanced Topics in Geometry E1 (MTH.B505)

Riemannian connection for submanifolds

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Exercise 3-1

Problem (Ex. 3-1)

Let $\mathrm{O}(2,1)$ *be the set of* 3×3*-matrices satisfying*

$$\mathrm{O}(2,1) := \left\{ A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \in \mathrm{M}_3(\mathbb{R})\,;\; A^T Y A = Y \right\}$$

$$\left(Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

Exercise 3-1

Problem (Ex. 3-1)

- *Show that* $|\det A| = 1$ *for* $A \in \mathrm{O}(2,1)$.
- *Show that* $|a_{00}| \geqq 1$ *for* $A = (a_{ij})$.
- *Show that the liner transformation induced by* $A \in \mathrm{O}(2,1)$ *preserves the inner product* $\langle\ ,\ \rangle$ *of* $\mathbb{E}^3_1$.
- $\mathrm{SO}_+(2,1) := \{A = (a_{ij}) \in \mathrm{O}(2,1)\,;\, \det A = 1, a_{00} \geqq 1\}$ *induces a bijection from the hyperbolic space* $H^2(k) \subset \mathbb{E}^3_1$ *to itself, where* $k < 0$.

Exercise 3-1

$$\mathrm{O}(n,1) := \{A \in \mathrm{M}_n(\mathbb{R})\,;\, A^T Y A = Y\}, \qquad Y = \mathrm{diag}(-1,1,\dots,1)$$

- $f\colon \mathbb{E}^{n+1}_1 \to \mathbb{E}^{n+1}_1$: a bijection preserving the Minkowski inner product $\Rightarrow f(\boldsymbol{x}) = A\boldsymbol{x}$ $(A \in \mathrm{O}(n,1))$
- $A \in \mathrm{O}(n,1) \Rightarrow \det A = \pm 1.$
- $A = (a_{ij}) \in \mathrm{O}(n,1) \Rightarrow |a_{00}| \geqq 1$

Exercise 3-2

Problem (Ex. 3-1)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$*, and set*

$$\boldsymbol{f}: D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3 = \mathbb{E}^3_1,$$

and take an orthonormal basis $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *of* $T_{\boldsymbol{x}}H^3(-1)$*, where* $\boldsymbol{x} = \boldsymbol{f}(u,v)$*.*

- *Verify that, for each* $(u,v) \in D$*,* $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$ *is a basis of* $\mathbb{R}^3$*, where* $\boldsymbol{e}_0 = \boldsymbol{f}$*.*
- *Express the derivatives* $(\boldsymbol{e}_j)_u$ *and* $(\boldsymbol{e}_j)_v$ $(j = 0, 1, 2)$ *as linear combinations of* $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$*.*