Problems on Lecture Note 4

⇒ revised version will be uploaded

Advanced Topics in Geometry E1 (MTH.B505)

Riemannian connection for submanifolds

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later use

- Lie brackets

(def'n) →

- Riemannian connection of submanifolds
- geodesics

Review

$M \qquad T_pM$

$\Big(X : p \longmapsto X_p \Big)$

- M: an n-dimensional manifold.
- T_pM: the tangent space of M at p.
- $\mathcal{F}(M)$: the algebra of C^∞-functions on M.
- $\mathfrak{X}(M)$: $\mathcal{F}(M)$-module of C^∞-vector fields on M.

Definition

A tangent vector of M at p is an $\mathbb{R}$-linear map $X_p : \mathcal{F}(M) \to \mathbb{R}$ satisfying the "Leibniz rule"

"directional derivative"

$$(X_p)(fg) = f(p)X_p(g) + g(p)X_p(f).$$

$X \in \mathfrak{X}(M)$, $f \in \mathcal{F}(M) \Rightarrow Xf \in \mathfrak{X}(M)$.

$\mathcal{F}$

Lie bracket

$(x^1, \cdots x^n)$: a local coordinate sys

local frame of vect. field

$$T_pM = \mathrm{Span}\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \cdots, \left(\frac{\partial}{\partial x^n}\right)_p\right\}$$

$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}, \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j} \quad \Rightarrow \quad Xf = \sum_{l=1}^{n} X^l \frac{\partial f}{\partial x^l},$$

$$Y(Xf) = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}\left(\sum_{l=1}^{n} X^l \frac{\partial f}{\partial x^l}\right)$$

$$= \sum_{j,l=1}^{n} Y^j \left(X^l \frac{\partial^2 f}{\partial x^j \partial x^l} + \frac{\partial X^l}{\partial x^j}\frac{\partial f}{\partial x^l}\right)$$

$$\boxed{\left(\frac{\partial}{\partial x^i}\right)_p f = \frac{\partial f}{\partial x^i}(p)}$$

$$X = \sum_i X^i \frac{\partial}{\partial x^i} \Rightarrow Xf = \sum X^i \frac{\partial f}{\partial x^i} \in \mathcal{F}(M)$$

functions in $(x^1 \cdots x^n)$

$$Y = \sum_i Y^i \frac{\partial}{\partial x^i} \qquad f \longmapsto Y(Xf)$$

not a tangent vector

$$\Rightarrow Y(Xf) = \sum_{i,j} Y^i \frac{\partial}{\partial x^i}\left(X^j \frac{\partial f}{\partial x^j}\right)$$

$$= \sum_{i,j}\left(Y^i X^j \frac{\partial^2 f}{\partial x^i \partial x^j} + Y^i \frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^j}\right)$$

Lie bracket

$X, Y \in \mathfrak{X}(M)$.

$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}, \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j} \quad \Rightarrow \quad X(Yf) - Y(Xf) = \sum_{j,l=1}^{n} \left(\right.$$

$$\sum \left(X^j \frac{\partial Y^l}{\partial x^j} - Y^j \frac{\partial X^l}{\partial y^j} \right) \frac{\partial f}{\partial x^l}$$

Definition

$[X, Y]$

The $\boxed{\text{Lie bracket}}$ of X and Y is defined as

$$[X, Y]f := X(Yf) - Y(Xf) \qquad (f \in \mathcal{F}(M)).$$

$$[X, Y] := \sum_{j,l} \left(X^j \frac{\partial Y^l}{\partial x^j} - Y^j \frac{\partial X^l}{\partial x^j} \right) \frac{\partial}{\partial x^l}$$

Lie bracket

$\mathfrak{X}(M)$: $\mathcal{F}(M)$-module

Lemma

For X, Y, $Z \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$, it hold that

- $[X, Y] = -[Y, X]$,
- $[fX, Y] = f[X, Y] - (Yf)X$, $[X, fY] = f[X, Y] + (Xf)Y$,
- $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$.

Jacobi's identity

→ symmetricity of the curvature tensor

R

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

☆ commutativity of partial differentiation

The Lie bracket as an integrability condition

Fact

domain

- $[X_1, \ldots, X_n]$: *an n-tuple of vector fields on $U \subset M$*
- $\{X_1, \ldots, X_n\}$ *are linearly independent on each point p.*

$\Rightarrow$

- $\exists$*a local coordinate system* $(x^1, \ldots, x^n)$ *around p with*

$$\left(X_j = \frac{\partial}{\partial x^j} \right) \qquad \cancel{X_j = \frac{\partial}{\partial x^j}} \; \frac{\partial}{\partial x^j} \; (j = 1, \ldots, n)$$

if and only if $\boxed{[X_j, X_k] = \mathbf{0}}$ *for all* $\underline{j, k = 1, \ldots, n}$.

a special case of Frobenius' thm

(proved in 2Q)

The canonical connection on $\mathbb{R}^n$

point   vectors

$X \in \mathfrak{X}(\mathbb{R}^n)$ can be considered $X = (X^1, \ldots, X^n)^T \colon \mathbb{R}^n \to \mathbb{R}^n$.

smooth functions

Definition

directional derivative

For $\boldsymbol{v} \in T_p\mathbb{R}^n$,

$$D_{\boldsymbol{v}} X := (dX^1(\boldsymbol{v}), \ldots, dX^n(\boldsymbol{v}))^T.$$

$(\boldsymbol{v}X^1, \ldots, \boldsymbol{v}X^n)^T$

Definition

The canonical connection of $\mathbb{R}^n$ is

$$D \colon \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \ni (X, Y) \mapsto D_X Y \in \mathfrak{X}(M).$$

Lemma

$$D_X Y - D_Y X = [X, Y]$$

Position vector field

$X \in \mathfrak{X}(\mathbb{R}^n)$ can be considered $X = (X^1, \ldots, X^n)^T \colon \mathbb{R}^n \to \mathbb{R}^n$.

Example

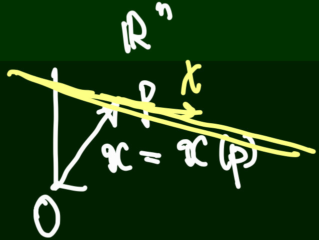
$\boldsymbol{x}$: the position vector field on $\mathbb{R}^n$ $\Rightarrow$

$$D_X \boldsymbol{x} = X$$

for $X \in \mathfrak{X}(\mathbb{R}^n)$.

$$D_X x = \frac{d}{dt}\Big|_{t=0} x(p + tX)$$

$$= X$$



Tangent space of a submanifold of $\mathbb{E}^N$

$M \subset \mathbb{E}^{n+r}$: an n-dimensional submanifold of the Euclidean space.

$$T_{\boldsymbol{x}}\mathbb{E}^{n+r} = \mathbb{E}^{n+r} = \underline{T_{\boldsymbol{x}}M} \oplus N_{\boldsymbol{x}} \qquad N_{\boldsymbol{x}} = (T_{\boldsymbol{x}}M)^{\perp}$$

$$\boxed{v = [v]^{\mathrm{T}} + [v]^{\mathrm{N}}} \qquad v \in T_{\boldsymbol{x}}\mathbb{E}^{n+r}$$

direct sum

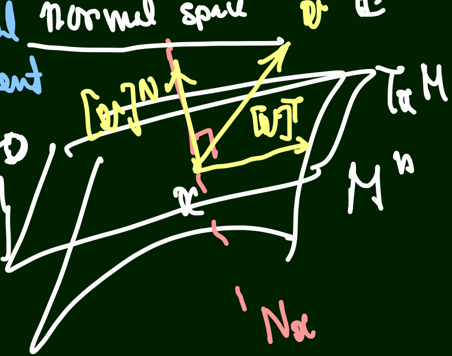
normal

tangential component

$N_x = (T_x M)^{\perp}$

$= \{ v \in \mathbb{E}^{n+r} \mid \langle v, w \rangle = 0$ for $\forall\, T_x M \}$

normal space



Induced connection

$M \subset \mathbb{E}^{n+r}$: an n-dimensional submanifold of the Euclidean space with induced metric.

Definition

∇: (nabla)

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \boxed{\nabla_X Y} := [D_X Y]^{T} \in \mathfrak{X}(M)$$

induced connection on $(M, \langle\, , \rangle)$

covariant derivative

∇ is an "intrinsic" notion

Next week

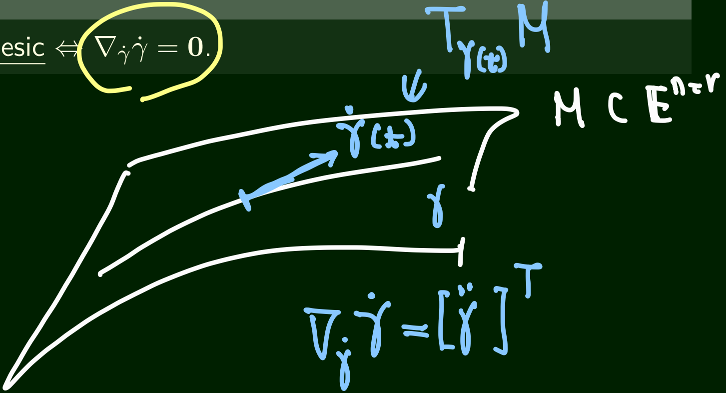
Geodesic

$M \subset \mathbb{E}^{n+r}$: an n-dimensional submanifold of the Euclidean space with induced metric.
∇: the induced connection.
$\gamma(t)$: a curve on M.

"straight line"

Definition

γ is a <u>geodesic</u> $\Leftrightarrow$ $\nabla_{\dot\gamma}\dot\gamma = \mathbf{0}$.



$$\nabla_{\dot\gamma}\dot\gamma = [\ddot\gamma]^T$$

Example: the unit sphere

$T_x S^n(1) = x^\perp$

$S^n(1) := \{x \in \mathbb{E}^{n+1} ; \langle x, x \rangle = 1\}$

$$\mathbb{E}^{n+1} = T_x\mathbb{E}^{n+1} = T_x S^n(1) \oplus \mathbb{R}x.$$

- $\gamma(t)$: a curve in $S^n(1)$

$$\nabla_{\dot\gamma}\dot\gamma := [\ddot\gamma]^T = \ddot\gamma - \langle \ddot\gamma, x \rangle x$$

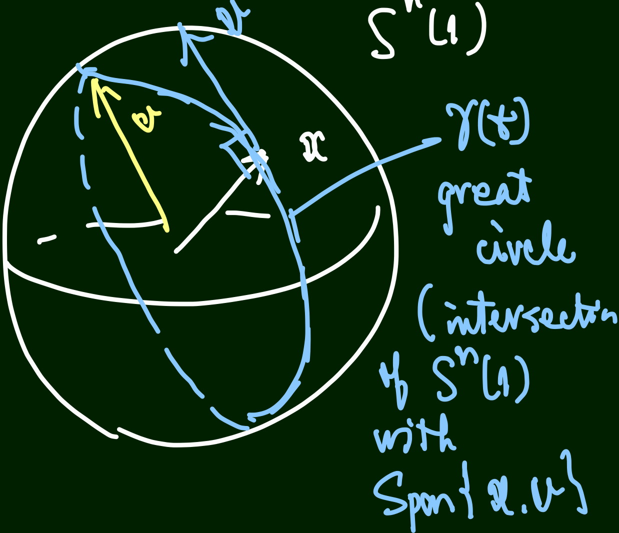
$x = \gamma$, $x = \gamma$

- $x \in S^n(1)$, $v \in T_x S^n(1)$ $x^\perp v$, $|v| = 1$

Set $\gamma(t) := (\cos t)x + (\sin t)v \in S^n(1)$

$\ddot\gamma(t) = -\gamma(t)$ $\boxed{\nabla_{\dot\gamma}\dot\gamma = 0}$

$\gamma(t) = \cos t\, x + \sin t\, v$: a geodesic



Exercise 4-1

Set

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1, x_0 > 0\}.$$

Problem (Ex. 4-1)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2 + v^2 < 1\}$, *and set*

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\big(1+u^2+v^2, 2u, 2v\big) \in H^3(-1)$$

and take an orthonormal frame $[\boldsymbol{e}_0(u,v), \underline{\boldsymbol{e}_1(u,v)}, \underline{\boldsymbol{e}_2(u,v)}]$.

- *Compute the Lie bracket* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *as a liner combination of* $\boldsymbol{e}_0$, $\boldsymbol{e}_1$ *and* $\boldsymbol{e}_2$.
  canonical connection of $\mathbb{R}^3 = \mathbb{E}^3_1$
- *Compute* $D_{\boldsymbol{e}_i}\boldsymbol{e}_j$ *for* i, $j = 1, 2$.

Exercise 4-2 Geodesics in $H^2(-1)$

Set

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1 \; \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}.$$

Problem (Ex. 4-2)

- *For each $\boldsymbol{x} \in H^2(-1)$, show that* $\boldsymbol{x}^\perp$

$$\mathbb{E}^3_1 = \boxed{T_{\boldsymbol{x}} H^2(-1)} \oplus \underline{\mathbb{R}\boldsymbol{x}}. \qquad (*)$$

- *Let $\boldsymbol{x} \in H^2(-1)$ and take a unit vector $\boldsymbol{v} \in T_{\boldsymbol{x}} H^2(-1) = \boldsymbol{x}^\perp$. Then show that*

geodesic of $H^2(-1)$

$$\underline{\gamma(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}} \in H^2(-1)$$

is a curve on $H^2(-1)$ satisfying $[\ddot{\gamma}(t)]^{\mathrm{T}} = \mathbf{0}$, where $[*]^{\mathrm{T}}$ denotes the $T_{\gamma(t)} H^2(-1)$-components of the decomposition $(*)$ with