Advanced Topics in Geometry E1 (MTH.B505)

Riemannian connection for submanifolds

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Review

- M: an n-dimensional manifold.
- T_pM: the tangent space of M at p.
- $\mathcal{F}(M)$: the algebra of C^∞-functions on M.
- $\mathfrak{X}(M)$: $\mathcal{F}(M)$-module of C^∞-vector fields on M.

Definition

A tangent vector of M at p is an $\mathbb{R}$-linear map $X_p \colon \mathcal{F}(M) \to \mathbb{R}$ satisfying the "Leibniz rule"

$$(X_p)(fg) = f(p)X_p(g) + g(p)X_p(f).$$

$X \in \mathfrak{X}(M)$, $f \in \mathcal{F}(M) \Rightarrow Xf \in \mathfrak{X}(M)$.

Lie bracket

$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}, \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j} \quad \Rightarrow \quad Xf = \sum_{l=1}^{n} X^l \frac{\partial f}{\partial x^l},$$

$$\begin{aligned} Y(Xf) &= \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j} \left(\sum_{l=1}^{n} X^l \frac{\partial f}{\partial x^l} \right) \\ &= \sum_{j,l=1}^{n} Y^j \left(X^l \frac{\partial^2 f}{\partial x^j \partial x^l} + \frac{\partial X^l}{\partial x^j} \frac{\partial f}{\partial x^l} \right) \end{aligned}$$

Lie bracket

X, $Y \in \mathfrak{X}(M)$.

$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}, \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j} \quad \Rightarrow \quad X(Yf) - Y(Xf) \quad = \sum_{j,l=1}^{n} \Bigg(X^j$$

Definition

The Lie bracket of X and Y is defined as

$$[X, Y]f := X(Yf) - Y(Xf) \qquad (f \in \mathcal{F}(M)).$$

Lie bracket

Lemma

For X, Y, $Z \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$, it hold that

- $[X, Y] = -[Y, X]$,
- $[fX, Y] = f[X, Y] - (Yf)X$, $[X, fY] = f[X, Y] + (Xf)Y$,
- $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$.

The Lie bracket as an integrability condition

Fact

- $[X_1, \ldots, X_n]$: *an* n*-tuple of vector fields on* $U \subset M$
- $\{X_1, \ldots, X_n\}$ *are linearly independent on each point* p*.*

$\Rightarrow$

- $\exists$*a local coordinate system* $(x^1, \ldots, x^n)$ *around* p *with*

$$X_j = \frac{\partial}{/}\partial x^j \qquad (j = 1, \ldots, n)$$

if and only if $[X_j, X_k] = \mathbf{0}$ *for all* $j, k = 1, \ldots, n$*.*

The canonical connection on $\mathbb{R}^n$

$X \in \mathfrak{X}(\mathbb{R}^n)$ can be considered $X = (X^1, \ldots, X^n)^T \colon \mathbb{R}^n \to \mathbb{R}^n$.

Definition

For $\boldsymbol{v} \in T_p\mathbb{R}^n$,

$$D_{\boldsymbol{v}} X := (dX^1(\boldsymbol{v}), \ldots, dX^n(\boldsymbol{v}))^T.$$

Definition

The canonical connection of $\mathbb{R}^n$ is

$$D \colon \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \ni (X, Y) \colon \to D_X Y \in \mathfrak{X}(M).$$

Lemma

$$D_X Y - D_Y X = [X, Y]$$

Position vector field

$X \in \mathfrak{X}(\mathbb{R}^n)$ can be considered $X = (X^1, \ldots, X^n)^T \colon \mathbb{R}^n \to \mathbb{R}^n$.

Example

$\boldsymbol{x}$: the position vector field on $\mathbb{R}^n \Rightarrow$

$$D_X \boldsymbol{x} = X$$

for $X \in \mathfrak{X}(\mathbb{R}^n)$.

Tangent space of a submanifold of $\mathbb{E}^N$

$M \subset \mathbb{E}^{n+r}$: an n-dimensional submanifold of the Euclidean space.

$$T_{\boldsymbol{x}}\mathbb{E}^{n+r} = \mathbb{E}^{n+r} = T_{\boldsymbol{x}}M \oplus N_{\boldsymbol{x}} \qquad N_{\boldsymbol{x}} = (T_{\boldsymbol{x}}M)^{\perp}$$

$$\boldsymbol{v} = [\boldsymbol{v}]^{\mathrm{T}} + [\boldsymbol{v}]^{\mathrm{N}} \qquad \boldsymbol{v} \in T_{\boldsymbol{x}}\mathbb{E}^{n+r}$$

Induced connection

$M \subset \mathbb{E}^{n+r}$: an n-dimensional submanifold of the Euclidean space with induced metric.

Definition

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y := [D_X Y]^{\mathrm{T}} \in \mathfrak{X}(M)$$

Geodesic

$M \subset \mathbb{E}^{n+r}$: an n-dimensional submanifold of the Euclidean space with induced metric.
∇: the induced connection.
$\gamma(t)$: a curve on M.

Definition

γ is a <u>geodesic</u> $\Leftrightarrow \nabla_{\dot{\gamma}}\dot{\gamma} = \mathbf{0}$.

Example: the unit sphere

$S^n(1) := \{\boldsymbol{x} \in \mathbb{E}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$

$$\mathbb{E}^{n+1} = T_{\boldsymbol{x}}\mathbb{E}^{n+1} = T_{\boldsymbol{x}}S^n(1) \oplus \mathbb{R}\boldsymbol{x}.$$

Exercise 4-1

Set

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}.$$

Problem (Ex. 4-1)

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2 < 1\}$, *and set*

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in H^3(-1)$$

and take an orthonormal frame $[\boldsymbol{e}_0(u,v), \boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$.

- *Compute the Lie bracket* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *as a liner combination of* $\boldsymbol{e}_0$, $\boldsymbol{e}_1$ *and* $\boldsymbol{e}_2$.
- *Compute* $D_{\boldsymbol{e}_i}\boldsymbol{e}_j$ *for* i, $j = 1, 2$.

Exercise 4-2

Set

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}.$$

Problem (Ex. 4-2)

- *For each* $\boldsymbol{x} \in H^2(-1)$*, show that*

$$\mathbb{E}^3_1 = T_{\boldsymbol{x}} H^2(-1) \oplus \mathbb{R}\boldsymbol{x}. \qquad (*)$$

- *Let* $\boldsymbol{x} \in H^2(-1)$ *and take a unit vector* $\boldsymbol{v} \in T_{\boldsymbol{x}} H^2(-1) = \boldsymbol{x}^\perp$*. Then show that*

$$\gamma(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$$

is a curve on $H^2(-1)$ *satisfying* $[\ddot{\gamma}(t)]^{\mathrm{T}} = \mathbf{0}$*, where* $[*]^{\mathrm{T}}$ *denotes the* $T_{\gamma(t)} H^2(-1)$*-components of the decomposition* $(*)$ *with* $\boldsymbol{x} = \gamma(t)$*.*