Advanced Topics in Geometry E1 (MTH.B505)

Riemannian connection

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Review

- M: an n-dimensional manifold.
- T_pM: the tangent space of M at p.
- $\mathcal{F}(M)$: the algebra of C^∞-functions on M.
- $\mathfrak{X}(M)$: $\mathcal{F}(M)$-module of C^∞-vector fields on M.

Definition

A tangent vector of M at p is an $\mathbb{R}$-linear map $X_p\colon \mathcal{F}(M) \to \mathbb{R}$ satisfying the "Leibniz rule"

$$(X_p)(fg) = f(p)X_p(g) + g(p)X_p(f).$$

$X \in \mathfrak{X}(M)$, $f \in \mathcal{F}(M) \Rightarrow Xf \in \mathfrak{X}(M)$.

Definition

The Lie bracket of X and Y is defined as

$$[X,Y]f := X(Yf) - Y(Xf) \qquad (f \in \mathcal{F}(M)).$$

Exercises 4

the hyperbolic plane

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1\,;\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x^0 > 0\}.$$

Exercise 4-1

Problem (Ex. 4-1)

$f = (x^0, x^1, x^2)^T$

$\langle f, f \rangle$

$= -(x^0)^2 + (x^1)^2 + (x^2)^2$

$= -1$

Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\}$, *and set*

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}(1+u^2+v^2, 2u, 2v)^T \in \underline{H^2(-1)}$$

and take an orthonormal frame $[\boldsymbol{e}_0(u,v), \boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$.

- *Compute the Lie bracket* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$.
- *Compute* $D_{\boldsymbol{e}_i}\boldsymbol{e}_j$ *for* i, $j = 1, 2$.

canonical connection on E^3_1
(the directional derivative)

Exercise 4-1

$\leftarrow \mathbb{H}^3_1$

$$\boldsymbol{f}(u,v) = \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right)$$

$$\boldsymbol{f}_u = \frac{2}{(1-u^2-v^2)^2}(2u, 1+u^2-v^2, 2uv)$$

$$\boldsymbol{f}_v = \frac{2}{(1-u^2-v^2)^2}(2v, 2uv, 1-u^2+v^2)$$

$$\langle f_u, f_u\rangle = \frac{4}{(\quad)^4}\left(-4u^2 + (1+u^2-v^2)^2 + 4u^2v^2\right)$$

$$= \frac{4}{(\quad)^4}(1-u^2-v^2)^2 = \frac{4}{(\quad)^2}$$

$$= \langle f_v, f_v\rangle$$

$$\langle f_u, f_v\rangle = 0, \quad \langle f, f_u\rangle = \langle f, f_v\rangle = 0$$

⟨e₀ e₀⟩ = −1

$$\boldsymbol{e}_0 = \boldsymbol{f} = \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right)$$

$$\boldsymbol{e}_1 = \frac{1-u^2-v^2}{2}\boldsymbol{f}_u = \frac{1}{1-u^2-v^2}\left(2u, 1+u^2-v^2, 2uv\right)$$

$\langle e_1, e_1\rangle = 1$

$$\boldsymbol{e}_2 = \frac{1-u^2-v^2}{2}\boldsymbol{f}_v = \frac{1}{1-u^2-v^2}\left(2v, 2uv, 1-u^2+v^2\right)$$

$\langle e_2, e_2\rangle = 1$

$\langle e_1, e_2\rangle = 0$

$[e_0\ \underbrace{e_1\ e_2}]$: an orthonormal frame

Spans the tangent plane of $H^2(-1)$ at $f(u,v)$

Exercise 4-1

in a parameter (u, v)

$$e_0 = f = \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right)$$

$$e_1 = \frac{1-u^2-v^2}{2} f_u = \frac{1}{1-u^2-v^2}\left(2u, 1+u^2-v^2, 2uv\right)$$

$$e_2 = \frac{1-u^2-v^2}{2} f_v = \frac{1}{1-u^2-v^2}\left(2v, 2uv, 1-u^2+v^2\right)$$

$\boxed{\varphi = \varphi(u, v)}$ a function on $H^2(-1)$

$e_1 \varphi = \left.\frac{d}{dt}\right|_{t=0} \varphi \circ \gamma$

$= \frac{1-u^2-v^2}{2} \varphi_u$

$\gamma(t)$: a curve on (u, v) plane

$\gamma(0) = (u, v)$ $\frac{1-u^2-v^2}{2} f_u = e_1$

$(f \circ \gamma)'(0) = e_1$

e.g. $\gamma(t) = \left(u + t\frac{1-u^2-v^2}{2}, v\right)$

Exercise 4-1

$$
\begin{aligned}
\boldsymbol{e}_0 &= \boldsymbol{f} &&= \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right)\\
\boldsymbol{e}_1 &= \frac{1-u^2-v^2}{2}\boldsymbol{f}_u &&= \frac{1}{1-u^2-v^2}\left(2u, 1+u^2-v^2, 2uv\right)\\
\boldsymbol{e}_2 &= \frac{1-u^2-v^2}{2}\boldsymbol{f}_v &&= \frac{1}{1-u^2-v^2}\left(2v, 2uv, 1-u^2+v^2\right)
\end{aligned}
$$

$$e_1\varphi = \frac{1-u^2-v^2}{2}\varphi_u$$

$$e_2\cdot(e_1\varphi) = \frac{1-u^2-v^2}{2}(e_1\varphi)_v = \frac{1-u^2-v^2}{2}\left(-2v\varphi_u + \frac{1-u^2-v^2}{2}\varphi_{uv}\right)$$

$$e_1\cdot(e_2\varphi) = \frac{1-u^2-v^2}{2}\left(\frac{1-u^2-v^2}{2}\varphi_{uv} - 2u\varphi_v\right)$$

$$e_1(e_2\varphi) - e_2(e_1\varphi) = \frac{1-u^2-v^2}{2}\left(v\varphi_u - u\varphi_v\right)$$

$$= v\, e_1\varphi - u\, e_2\varphi$$

$$= (v e_1 - u e_2)\varphi$$

$$[e_1\ e_2] = v e_1 - u e_2\ ,$$

$$e_j = e_j(u,v)$$

$$e_1 = \frac{1-u^2-v^2}{2}\partial_u$$

$$D_{e_1} e_1 = \frac{1-u^2-v^2}{2} (e_1)_u$$

$$= \underbrace{\frac{1-u^2-v^2}{2}}\,\frac{2}{1-u^2-v^2}\left(e_0 - v e_2\right) = e_0 - v e_2$$

Similarly

$$D_{e_1} e_2 = v e_1 \qquad D_{e_2} e_1 = u e_2$$

$$D_{e_2} e_2 = e_0 - u e_1$$

Rem "$D_{e_1} e_2 - D_{e_2} e_1 = v e_1 - u e_2 = [e_1, e_2]$"

"torsion free" property of D

Geodesic

Problem (Ex. 4-2)

- *For each* $\boldsymbol{x} \in H^2(-1)$, *show that*

$$\mathbb{E}^3_1 = T_{\boldsymbol{x}} H^2(-1) \oplus \mathbb{R}\boldsymbol{x}. \qquad (*)$$

- *Let* $\boldsymbol{x} \in H^2(-1)$ *and take a unit vector* $\boldsymbol{v} \in T_{\boldsymbol{x}} H^2(-1) = \boldsymbol{x}^\perp$. *Then show that*

$$\gamma(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$$

is a curve on $H^2(-1)$ *satisfying* $[\ddot{\gamma}(t)]^{\mathrm{T}} = \mathbf{0}$, *where* $[*]^{\mathrm{T}}$ *denotes the* $T_{\gamma(t)} H^2(-1)$*-component of the decomposition* ($*$) *with* $\boldsymbol{x} = \gamma(t)$.

$x \in H^2(-1) \subset \mathbb{E}^3_1$ $\qquad$ $\mathbb{E}^3_1 = T_x\mathbb{E}^3_1 = \underbrace{T_xH^2(-1)}_{\text{2-dim}} \oplus \underbrace{\mathbb{R}x}_{\text{1-dim}}$

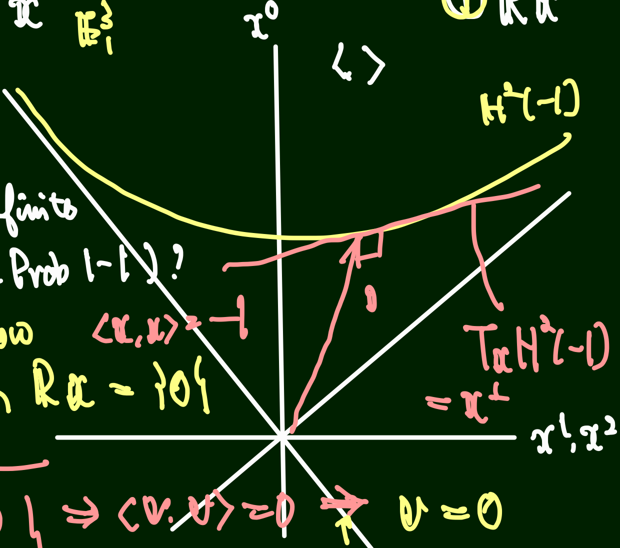
$T_xH^2(-1) = x^{\perp_{\mathbb{E}^3_1}}$

- $\langle\, , \rangle|_{TH^2(-1)}$ is positive definite (Prob 1-1)?

- Sufficient to show $\boxed{T_xH^2(-1)} \cap \mathbb{R}x = \{0\}$

- $v \in \_\_\_\_ \cap \_\_\_\_$

$\rightarrow$ $\left.\begin{matrix} \langle v, v\rangle \geq 0 \\ \langle v, v\rangle \leq 0 \end{matrix}\right\} \Rightarrow \langle v, v\rangle = 0 \Rightarrow v = 0$



$T_xH^2(-1)$: positive

$$\mathbb{E}^3_1 = T_x\mathbb{E}^3_1 = \underline{T_x H^2(-1) \oplus \mathbb{R}x}$$ (cf. S^2-case)

- $x \in H^2(-1)$, $\quad v \in T_x H^2(-1) = x^\perp$
  with $\langle v, v \rangle = 1$

Set $\gamma(t) = \cosh t \cdot x + \sinh t \; v \in \mathbb{E}^3_1$

$$\langle \gamma(t), \gamma(t) \rangle = \cosh^2 t \, \overbrace{\langle x, x \rangle}^{=-1} + 2 \cosh t \sinh t \, \cancel{\langle x, v \rangle} + \sinh^2 t \, \overbrace{\langle v, v \rangle}^{=1}$$

$$= -\cosh^2 t + \sinh^2 t = -1$$

$\gamma(t) \in H^2(-1)$

- $\gamma(t)$: a curve on $H^2(-1)$
  with $\gamma(0) = x \qquad \dot\gamma(0) = v$

- $\ddot\gamma(t) = \cosh t \; x + \sinh t \; v$
  $= \gamma(t) \perp T_{\gamma(t)} H^2(-1)$

  $[\ddot\gamma(t)]^T = 0$

  - Riemannian connection
  - Induced connection

- We've not defined the "induced connection" for submanifolds of $\mathbb{E}^3_1$ (Lect. 05)
- γ is a geodesic of $H^2(-1)$