Advanced Topics in Geometry E1 (MTH.B505)

Riemannian connection

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2023/geom-e1/

Tokyo Institute of Technology

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Review

- M: an n-dimensional manifold.
- T_pM: the tangent space of M at p.
- $\mathcal{F}(M)$: the algebra of C^∞-functions on M.
- $\mathfrak{X}(M)$: $\mathcal{F}(M)$-module of C^∞-vector fields on M.

Definition

A tangent vector of M at p is an $\mathbb{R}$-linear map $X_p \colon \mathcal{F}(M) \to \mathbb{R}$ satisfying the "Leibniz rule"

$$(X_p)(fg) = f(p)X_p(g) + g(p)X_p(f).$$

$X \in \mathfrak{X}(M)$, $f \in \mathcal{F}(M) \Rightarrow Xf \in \mathfrak{X}(M)$.

Definition

The Lie bracket of X and Y is defined as

$$[X,Y]f := X(Yf) - Y(Xf) \qquad (f \in \mathcal{F}(M)).$$

Exercises 4

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x^0 > 0\}.$$

Exercise 4-1

Problem (Ex. 4-1)

Let $D := \{(u,v) \in \mathbb{R}^2 \,;\, u^2 + v^2 < 1\}$, *and set*

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in H^2(-1)$$

and take an orthonormal frame $[\boldsymbol{e}_0(u,v), \boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$.

- *Compute the Lie bracket* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$.
- *Compute* $D_{\boldsymbol{e}_i}\boldsymbol{e}_j$ *for* i, $j = 1, 2$.

Exercise 4-1

$$\boldsymbol{f}(u,v) = \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right)$$
$$\boldsymbol{f}_u = \frac{2}{1-u^2-v^2}\left(2u, 1+u^2-v^2, 2uv\right)$$
$$\boldsymbol{f}_v = \frac{2}{1-u^2-v^2}\left(2v, 2uv, 1-u^2+v^2\right)$$

Exercise 4-1

$$
\begin{aligned}
\boldsymbol{e}_0 = \boldsymbol{f} \qquad\qquad &= \frac{1}{1-u^2-v^2}\left(1+u^2+v^2, 2u, 2v\right) \\
\boldsymbol{e}_1 = \frac{1-u^2-v^2}{2}\boldsymbol{f}_u &= \frac{1}{1-u^2-v^2}\left(2u, 1+u^2-v^2, 2uv\right) \\
\boldsymbol{e}_2 = \frac{1-u^2-v^2}{2}\boldsymbol{f}_v &= \frac{1}{1-u^2-v^2}\left(2v, 2uv, 1-u^2+v^2\right)
\end{aligned}
$$

Exercise 4-2

Problem (Ex. 4-2)

- *For each* $\boldsymbol{x} \in H^2(-1)$, *show that*

$$\mathbb{E}^3_1 = T_{\boldsymbol{x}} H^2(-1) \oplus \mathbb{R}\boldsymbol{x}. \qquad (*)$$

- *Let* $\boldsymbol{x} \in H^2(-1)$ *and take a unit vector* $\boldsymbol{v} \in T_{\boldsymbol{x}} H^2(-1) = \boldsymbol{x}^\perp$. *Then show that*

$$\gamma(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$$

is a curve on $H^2(-1)$ *satisfying* $[\ddot{\gamma}(t)]^{\mathrm{T}} = \mathbf{0}$, *where* $[*]^{\mathrm{T}}$ *denotes the* $T_{\gamma(t)} H^2(-1)$*-component of the decomposition* $(*)$ *with* $\boldsymbol{x} = \gamma(t)$.