- M: a (pseudo) Riemannian manifold with metric $g (= \langle\ ,\ \rangle)$

Advanced Topics in Geometry E1 (MTH.B505)

Riemannian connection

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$\langle\ ,\ \rangle$: non-degenarate $\Leftrightarrow$ $\boxed{\begin{array}{l}(\langle x, v \rangle = 0 \text{ for } \forall v) \\ \Leftrightarrow x = 0\end{array}}$

Riemannian connection

for an abstract Riem. manifold (not nec. submfd of $\mathbb{E}^n$).

Lemma (Lemma 5.1)

There exists the unique bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \longmapsto \nabla_X Y \in \mathfrak{X}(M)$$

nabla

$\mathfrak{F}(M) = \{\text{smooth functions}\}$

$\mathfrak{X}(M) := \{\text{vector fields}\}$

satisfying

- $\nabla_X Y - \nabla_Y X = [X, Y]$,
- $X \underbrace{\langle Y, Z \rangle}_{\in \mathfrak{F}(M)} = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$.

for $X, Y, Z \in \mathfrak{X}(M)$.

Lebnitz rule for $\langle\, ,\, \rangle$

"torsion free" condition

∇ : the Riemannian connection

the Levi-Civita

Lemma 5.1

$$2C(X,Y,Z) := X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle + \langle X,[Z,Y]\rangle - \langle Z,[X,Y]\rangle$$

$\langle \nabla_X Y, Z\rangle$ $\qquad$ $Y\ X$

If $\exists\, \nabla$,

$$\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle - \langle Y, \nabla_X Z\rangle$$
$$= X\langle Y,Z\rangle - \langle Y, \nabla_Z X + [X,Z]\rangle$$
$$= X\langle Y,Z\rangle - \langle Y, \nabla_Z X\rangle - \langle Y,[X,Z]\rangle$$
$$= X\langle Y,Z\rangle - Z\langle Y, X\rangle + \langle \nabla_Z Y, X\rangle - \langle Y,[X,Z]\rangle$$
$$= X\langle Y,Z\rangle - Z\langle Y, X\rangle + \langle \nabla_Y Z + [Z,Y], X\rangle$$
$$= X\langle Y,Z\rangle - Z\langle Y, X\rangle \quad \text{"} \nabla_X Y + [Y,X] \quad - \langle Y,[X,Z]\rangle$$
$$+ Y\langle Z,X\rangle - \langle Z, \nabla_Y X\rangle + \langle [Z,Y], X\rangle$$
$$- \langle Y,[X,Z]\rangle$$

$\nabla_X Y :=$ a vector field satisfying

$\nabla_0 Y$

$$\langle \nabla_X Y, Z \rangle = \boxed{C(X, Y, Z)}.$$

- uniqueness follows because of non degeneracy of $\langle , \rangle$.

- existence :

$\Big($ $\varphi : V \to \mathbb{R}$: linear, $\langle , \rangle$ non-degenerate on V

$\Rightarrow \exists!\, v \in V$ s.t. $\varphi(x) = \langle v, x \rangle$. $\Big)$

Definition

The map ∇ in Lemma 5.1 is called the Riemannian connection or the Levi-Civita connection.

What is connection? (many meaning)

- Linear connection on tangent bundle TM
- $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ bilinear

  satisfying ($\to$ Lemma on next page)

Riemannian (Levi-Civita) connection

$f \in \mathfrak{F}(M)$ $X, Y \in \mathfrak{X}(M)$

Lemma

- $\boxed{\nabla_{fX} Y = f \nabla_X Y}$ ($\mathfrak{F}(M)$-linear in X) $\{X \in \mathfrak{X}(M)$
- $\nabla_X(\underbrace{fY}) = f\nabla_X Y + (Xf)Y$ scalar multiplic.

"Leibniz rule"

"($\exists$ many linear connections on TM)"

"gauge theory" is a kind of geometry on the space of connections.

Covariant Derivative

$[e_1 \cdots e_n]$

Corollary

Assume $X, Y \in \mathfrak{X}(M)$ *satisfy* $\boxed{X_p = Y_p}$ *at a point* $p \in M$. *Then*

$$(\nabla_X Z)_p = (\nabla_Y Z)_p$$

holds for each $Z \in \mathfrak{X}(M)$.

$X = \sum X^i e_i \quad Y = \sum Y^i e_i$

at p $\quad \nabla_X Z = \sum X^i (\nabla_{e_i} Z)_p = \sum Y^i (\nabla_{e_i} Z)_p$

Definition

For $\boldsymbol{x} \in T_pM$ be a tangent vector at $p \in M$ and a vector field $Y \in \mathfrak{X}(M)$,

$$\boxed{\nabla_{\boldsymbol{x}} Y := (\nabla_X Y)_p \in T_pM}$$

is called the covariant derivative of Y with respect to the direction $\boldsymbol{x}$, where $X \in \mathfrak{X}(M)$ is a vector field satisfying $X_p = \boldsymbol{x}$.

x

Example $\left(\mathbb{E}^{n+r}_r, \langle , \rangle\right)$ $\langle x, y\rangle = x^T \begin{pmatrix} -1 & & & & & 0 \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} y$

Independent of sign of $\langle , \rangle$

$$D\colon \mathfrak{X}(\mathbb{E}^{n+r}_r) \times \mathfrak{X}(\mathbb{E}^{n+r}_r) \ni (X, Y) \longmapsto D_X Y := \underline{dY(X)} \in \mathfrak{X}(M)$$

$\left(X \colon \mathbb{E}^{n+r}_r \longrightarrow \mathbb{E}^{n+r}_r\right)$

Then D is Levi-Civita connection on $\mathbb{E}^{n+r}_r$ (Lemma 4.6) (canonical connection)

Submanifolds

Lemma

Let $M \subset \mathbb{E}^{n+r}_r$ be a submanifold of the pseudo Euclidean space $\mathbb{E}^{n+r}_r$. If the restriction of the inner product $\langle\ ,\ \rangle$ of $\mathbb{E}^{n+r}_r$ on T_pM is non-degenerate, the direct sum decomposition

$$\mathbb{E}^{n+r}_r = T_pM \oplus (T_pM)^\perp,$$

that is, for each vector $\boldsymbol{v} \in \mathbb{E}^{n+r}_r = T_p\mathbb{E}^{n+r}_r$, there exists a unique decomposition

$$\boldsymbol{v} = [\boldsymbol{v}]^{\mathrm{T}} + [\boldsymbol{v}]^{\mathrm{N}}, \qquad [\boldsymbol{v}]^{\mathrm{T}} \in T_pM, \qquad [\boldsymbol{v}]^{\mathrm{N}} \in (T_pM)^\perp.$$

$M \subset \mathbb{E}^n \quad \mathbb{E}^n = T_p\mathbb{E}^n = \underbrace{T_pM}_{} \oplus \underbrace{(T_pM)^\perp}_{}$

$v = [v]^T + [v]^N$

$$\nabla_X Y := [D_X Y]^T$$

induced connection

(Thm) the Levi-Civita connection on $(M, \langle , \rangle|_{TM})$

$X, Y, Z \in \mathfrak{X}(M)$

☺ · $[D_X Y - D_Y X]^T = [[X, Y]]^T = [X, Y]$

$\quad \overset{\shortparallel}{\nabla_X Y - \nabla_Y X}$

tangent

· $X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$

$= \langle \underbrace{[D_X Y]^T}_{\nabla_X Y}, Z\rangle + \langle Y, \underbrace{[D_X Z]^T}_{\nabla_X Z}\rangle$

$M \subset \mathbb{E}^{n+r}_r$ Assumption: $\langle , \rangle|_{T_pM}$: $\boxed{\text{non deg. at } \forall p \in M}$

$\Rightarrow \quad \mathbb{E}^{n+r}_r = T_pM \oplus (T_pM)^\perp$

$\qquad\quad v = [v]^T + [v]^N$

$\Rightarrow \quad [D_X Y]^T$: the Levi-Civita connection

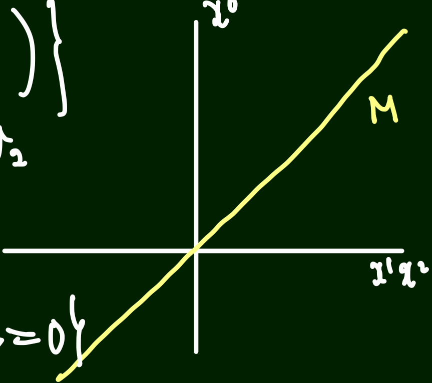
Example $\mathbb{E}^3_1 \supset M = \{ (\overset{x}{u}, u, v) \,;\, u, v \in \mathbb{R} \}$ x_u x_v

$$T_pM = \mathrm{Span}\left\{ \underset{v_1}{\underset{\shortparallel}{\begin{pmatrix}1\\1\\0\end{pmatrix}}}, \underset{v_2}{\underset{\shortparallel}{\begin{pmatrix}0\\0\\1\end{pmatrix}}} \right\}$$

$$(T_pM)^\perp \subset \mathbb{E}^3_1 : \text{1-dim}$$

$$\shortparallel$$

$$\{ w \mid \langle w, v_1 \rangle = \langle w, v_2 \rangle = 0 \}$$



$v_1 \in (T_pM)^\perp$ ✓ $\quad (T_pM)^\perp = \mathbb{R} \cdot v_1 \subset T_pM$

$$\boxed{\langle v_1, v_1 \rangle = 0}$$

Riemannian connection of submanifolds

Theorem

Let $M \subset \mathbb{E}^{n+r}_r$ be a submanifold such that the restriction of the inner product $\langle\ ,\ \rangle$ to TM is non-degenerate. We set for X, $Y \in \mathfrak{X}(M)$ by

$$\nabla_X Y := [D_X Y]^{\mathrm{T}}.$$

Then ∇ is the Levi-Civita connection of M with respect to the induced metric $\langle\ ,\ \rangle|_{TM}$.

Exercise

Set

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x^0 > 0\},$$

and take a parametrization

$$\boldsymbol{f} : D \ni (u, v) \mapsto \frac{1}{1 - u^2 - v^2}\bigl(1 + u^2 + v^2, 2u, 2v\bigr) \in H^2(-1)$$

of $H^2(-1)$, where $D := \{(u, v) \in \mathbb{R}^2 \,;\, u^2 + v^2 < 1\}$.

Exercise 5-1

Problem (Ex. 5-1)

Let $[\boldsymbol{e}_0(u,v), \boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *be an orthonormal frame as*

$$\boldsymbol{e}_0 := \boldsymbol{f}, \qquad \boldsymbol{e}_1 := \frac{\boldsymbol{f}_u}{|\boldsymbol{f}_u|}, \qquad \boldsymbol{e}_2 := \frac{\boldsymbol{f}_v}{|\boldsymbol{f}_v|}.$$

For the induced connection ∇ *of* $H^2(-1)$*,*

- *Compute* $\langle \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{e}_k \rangle$ *for* i*,* j *and* k *run over* $\{1,2\}$*.*
- *Compute* $\nabla_{\boldsymbol{e}_1}\nabla_{\boldsymbol{e}_2}\boldsymbol{e}_2 - \nabla_{\boldsymbol{e}_2}\nabla_{\boldsymbol{e}_1}\boldsymbol{e}_2 - \nabla_{[\boldsymbol{e}_1,\boldsymbol{e}_2]}\boldsymbol{e}_2$*.*

Exercise 5-2

Problem (Ex. 5-2)

Let $\widetilde{D} := (0,\infty)\times(-\pi,\pi)$,

$$\widetilde{\boldsymbol{f}} : \widetilde{D} \ni (r,t) \mapsto (\cosh r, \sinh r \cos t, \sinh r \sin t) \in H^2(-1)$$

and set

$$\boldsymbol{v}_0 = \widetilde{\boldsymbol{f}}, \qquad \boldsymbol{v}_1 = \widetilde{\boldsymbol{f}}_r, \qquad \boldsymbol{v}_2 = \frac{1}{\sinh r}\widetilde{\boldsymbol{f}}_t.$$

- *Find a parameter change* $\varphi\colon (r,t) \mapsto (u,v) = \big(u(r,t), v(r,t)\big)$.
- *Find a* 2×2*-matrix valued function* $\Theta = \Theta(r,t)$ *satisfying* $(\boldsymbol{e}_1,\boldsymbol{e}_2) = (\boldsymbol{v}_1,\boldsymbol{v}_2)\Theta$.