Advanced Topics in Geometry E1 (MTH.B505)

Riemannian connection

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Riemannian connection

Lemma (Lemma 5.1)

There exists the unique bilinear map

$$\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \longmapsto \nabla_X Y \in \mathfrak{X}(M)$$

satisfying

- $\nabla_X Y - \nabla_Y X = [X, Y]$,
- $X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$

for X, Y, $Z \in \mathfrak{X}(M)$.

Lemma 5.1

$$2C(X,Y,Z) := X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle + \langle X,[Z,Y]\rangle - \langle Z,[X,Y]\rangle$$

Riemannian (Levi-Civita) connection

Definition

The map ∇ in Lemma 5.1 is called the Riemannian connection or the Levi-Civita connection.

Riemannian (Levi-Civita) connection

Lemma

- $\nabla_{fX} Y = f \nabla_X Y$,
- $\nabla_X (fY) = f \nabla_X Y + (Xf) Y$

Covariant Derivative

Corollary

Assume X, $Y \in \mathfrak{X}(M)$ *satisfy* $X_p = Y_p$ *at a point* $p \in M$. *Then*

$$(\nabla_X Z)_p = (\nabla_Y Z)_p$$

holds for each $Z \in \mathfrak{X}(M)$.

Definition

For $\boldsymbol{x} \in T_pM$ be a tangent vector at $p \in M$ and a vector field $Y \in \mathfrak{X}(M)$,

$$\nabla_{\boldsymbol{x}} Y := (\nabla_X Y)_p \in T_pM$$

is called the covariant derivative of Y with respect to the direction $\boldsymbol{x}$, where $X \in \mathfrak{X}(M)$ is a vector field satisfying $X_p = \boldsymbol{x}$.

Example

$$D\colon \mathfrak{X}(\mathbb{E}^{n+r}_r) \times \mathfrak{X}(\mathbb{E}^{n+r}_r) \ni (X, Y) \longmapsto D_X Y := dY(X) \in \mathfrak{X}(M)$$

Submanifolds

Lemma

Let $M \subset \mathbb{E}^{n+r}_r$ *be a submanifold of the pseudo Euclidean space* $\mathbb{E}^{n+r}_r$*. If the restriction of the inner product* $\langle\ ,\ \rangle$ *of* $\mathbb{E}^{n+r}_r$ *on* T_pM *is non-degenerate, the direct sum decomposition*

$$\mathbb{E}^{n+r}_r = T_pM \oplus (T_pM)^\perp,$$

that is, for each vector $\boldsymbol{v} \in \mathbb{E}^{n+r}_r = T_p\mathbb{E}^{n+r}_r$*, there exists a unique decomposition*

$$\boldsymbol{v} = [\boldsymbol{v}]^{\mathrm{T}} + [\boldsymbol{v}]^{\mathrm{N}}, \qquad [\boldsymbol{v}]^{\mathrm{T}} \in T_pM, \qquad [\boldsymbol{v}]^{\mathrm{N}} \in (T_pM)^\perp.$$

Riemannian connection of submanifolds

Theorem

Let $M \subset \mathbb{E}_r^{n+r}$ *be a submanifold such that the restriction of the inner product* $\langle\ ,\ \rangle$ *to* TM *is non-degenerate. We set for* X*,* $Y \in \mathfrak{X}(M)$ *by*

$$\nabla_X Y := [D_X Y]^{\mathrm{T}}.$$

Then ∇ *is the Levi-Civita connection of* M *with respect to the induced metric* $\langle\ ,\ \rangle|_{TM}$*.*

Exercise

Set

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x^0 > 0\},$$

and take a parametrization

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1 - u^2 - v^2}\bigl(1 + u^2 + v^2, 2u, 2v\bigr) \in H^2(-1)$$

of $H^2(-1)$, where $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2 + v^2 < 1\}$.

Exercise 5-1

Problem (Ex. 5-1)

Let $[\boldsymbol{e}_0(u,v), \boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *be an orthonormal frame as*

$$\boldsymbol{e}_0 := \boldsymbol{f}, \qquad \boldsymbol{e}_1 := \frac{\boldsymbol{f_u}}{|\boldsymbol{f}_u|}, \qquad \boldsymbol{e}_2 := \frac{\boldsymbol{f_v}}{|\boldsymbol{f}_v|}.$$

For the induced connection ∇ *of* $H^2(-1)$*,*

- *Compute* $\langle \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{e}_k \rangle$ *for* i*,* j *and* k *run over* $\{1,2\}$*.*
- *Compute* $\nabla_{\boldsymbol{e}_1} \nabla_{\boldsymbol{e}_2} \boldsymbol{e}_2 - \nabla_{\boldsymbol{e}_2} \nabla_{\boldsymbol{e}_1} \boldsymbol{e}_2 - \nabla_{[\boldsymbol{e}_1, \boldsymbol{e}_2]} \boldsymbol{e}_2$*.*

Exercise 5-2

Problem (Ex. 5-2)

Let $\widetilde{D} := (0,\infty)\times(-\pi,\pi)$,

$$\widetilde{\boldsymbol{f}} : \widetilde{D} \ni (r,t) \mapsto (\cosh r, \sinh r \cos t, \sinh r \sin t) \in H^2(-1)$$

and set

$$\boldsymbol{v}_0 = \widetilde{\boldsymbol{f}}, \qquad \boldsymbol{v}_1 = \widetilde{\boldsymbol{f}}_r, \qquad \boldsymbol{v}_2 = \frac{1}{\sinh r}\widetilde{\boldsymbol{f}}_t.$$

- *Find a parameter change* $\varphi\colon (r,t)\mapsto (u,v) = \bigl(u(r,t), v(r,t)\bigr)$.
- *Find a* 2×2*-matrix valued function* $\Theta = \Theta(r,t)$ *satisfying* $(\boldsymbol{e}_1,\boldsymbol{e}_2) = (\boldsymbol{v}_1,\boldsymbol{v}_2)\Theta$.